A SHORT COURSE IN THE THEORY OF DETERMINANTS

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ \Delta^{(1)}a_1 & \Delta^{(1)}a_2 & \Delta^{(1)}a_3 & \cdots & \Delta^{(1)}a_n \\ \Delta^{(2)}a_1 & \Delta^{(2)}a_2 & \Delta^{(2)}a_3 & \cdots & \Delta^{(2)}a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta^{(n-1)}a_1 & \Delta^{(n-1)}a_2 & \Delta^{(n-1)}a_3 & \cdots & \Delta^{(n-1)}a_n \end{vmatrix}$$

LAENAS GIFFORD WELD

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THEORY OF DETERMINANTS,

BY

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New York:
MACMILLAN AND CO.
AND LONDON.
1893.

PREFACE.

THE aim of the author of the present work has been to develop the Theory of Determinants in the simplest possible manner. Great care has been taken to introduce the subject in such a way that any reader having an acquaintance with the principles of elementary Algebra can intelligently follow this development from the beginning. The last two chapters must be omitted by the student who is not familiar with the Calculus, and the same is to be said in reference to some few of the preceding articles; but in no case will the continuity of the course be affected by such omissions.

No attempt has been made to apply the theory to Analytical Geometry, though a few of its more important applications to Algebra have been included. The reader familiar with geometrical analysis will be interested in giving to the greater number of these applications, as also to many of the examples, their geometrical interpretations.

The earlier the student, in his mathematical course, is made familiar with the notation and methods of Determinants, the earlier will he be

prepared to appreciate the wonderful symmetry and generality so characteristic of the various modern developments in mathematics. In consideration of the limited time available for the study of such topics in the ordinary college course, the attempt has been made to render the book as readable as possible, rather than to prepare a drill book. However, it is hoped that the student who solves the two or three hundred examples proposed may thereby receive valuable mental discipline.

Acknowledgments are due the writings of *Muir*, *Scott*, *Hanus*, *Salmon*, *Baltzer*, *Günther*, *Clebsch*, *Mansion*, *Dostor*, *Houël*, and others. Even greater indebtedness to some of the above-named writers would doubtless have added much to whatever of merit the work now submitted may possess.

L. G. W.

IOWA CITY, IOWA, U.S.A.
1893, March 10th.

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CHAPTER I.

THE ORIGIN AND NOTATION OF DETERMINANTS.

In the various processes of analysis there are certain classes of functions which occur with remarkable frequency. Of these there are some which have been studied with great care, and of which general theories have been developed. The functions known as determinants constitute such a class, and one which is of great importance in some of the modern developments of mathematics.

One of the simplest of the many processes giving rise to determinants, and that which led Leibnitz in 1693 to their discovery, is the solution of systems of simultaneous equations of the first degree. In the present chapter we shall give examples of a few determinants resulting from this process, and explain the general principles of the notation usually employed in representing such functions.

1. Let us solve the two simultaneous linear equations,

$$a_1x + b_1y = \kappa_1,$$

$$a_2x + b_2y = \kappa_2.$$

Multiplying the first equation by b_2, the second by $-b_1$, and adding the resulting products, we readily obtain

$$x = \frac{\kappa_1 b_2 - \kappa_2 b_1}{a_1 b_2 - a_2 b_1}. \qquad (1)$$

Similarly,

$$y = \frac{a_1 \kappa_2 - a_2 \kappa_1}{a_1 b_2 - a_2 b_1}. \qquad (2)$$

These two fractions, which express the values of x and y, have a common denominator, which is a function of the coefficients of x and y in the given simultaneous equations. This function,

$$a_1 b_2 - a_2 b_1, \qquad (3)$$

is called the **determinant** of the coefficients a_1, b_1, a_2, and b_2, and in the notation of determinants it is commonly expressed by the symbol

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \qquad (3')$$

in which the coefficients are arranged in the same order as in the given equations.

This symbol is called a **determinant array.**

Of the determinant represented by the above array, a_1, b_1, a_2, and b_2 are called **elements** or **constituents.** The polynomial (3) is called the expanded form, or simply the **expansion**, of the determinant. Since each term of this expansion is the product of *two* elements, the determinant is said to be of the *second* **order.**

2. In the identical equation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1$$

the first member must be so interpreted that it shall represent the same function of a_1, b_1, a_2, and b_2 as the second member; that is:

The determinant array of the second order must be understood to mean that the product of the elements on the diagonal passing from the lower left-hand corner to the upper right-hand corner of the array is to be subtracted from the product of the elements on the other diagonal.

3. The numerators of the fractions in Equations (1) and (2) of Article 1 may also be written in the form of determinant arrays. Thus, by the preceding article,

$$\kappa_1 b_2 - \kappa_2 b_1 \equiv \begin{vmatrix} \kappa_1 & b_1 \\ \kappa_2 & b_2 \end{vmatrix},$$

$$a_1 \kappa_2 - a_2 \kappa_1 \equiv \begin{vmatrix} a_1 & \kappa_1 \\ a_2 & \kappa_2 \end{vmatrix}.$$

Expressed in the notation of determinants, the values of x and y in the given simultaneous equations are

$$x = \frac{\begin{vmatrix} \kappa_1 & b_1 \\ \kappa_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \text{ and } y = \frac{\begin{vmatrix} a_1 & \kappa_1 \\ a_2 & \kappa_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

NOTE. — The numerator of the fraction expressing the value of x may be formed from the denominator of the same fraction by replacing a_1 and a_2, the coefficients of x, by the absolute terms κ_1 and κ_2, respectively.

Similarly for y.

EXAMPLES.

1. Expand $\begin{vmatrix} 1 & x \\ x & 1 \end{vmatrix}$. *Ans.* $1 - x^2$.

2. Evaluate $\begin{vmatrix} 0 & -3 \\ 5 & 0 \end{vmatrix}$. *Ans.* 15.

3. Expand and reduce $\begin{vmatrix} \sin x & \sin y \\ \cos x & \cos y \end{vmatrix}$. *Ans.* $\sin(x - y)$.

4. Evaluate $\begin{vmatrix} 0 & 0 \\ a & b \end{vmatrix}$.

5. Evaluate $\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix}$.

6. Expand and reduce $\begin{vmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{vmatrix}$.

7. Expand and reduce $\begin{vmatrix} \cos x & \sin y \\ -\sin x & \cos y \end{vmatrix}$.

8. Expand and reduce $\begin{vmatrix} \log x & \log y \\ n & m \end{vmatrix}$. *Ans.* $\log \dfrac{x^m}{y^n}$.

9. Expand and reduce $\begin{vmatrix} 1 & -1 \\ 1 & -\cos x \end{vmatrix}$. *Ans.* $2\sin^2 \frac{1}{2}x$.

10. Expand and reduce $\begin{vmatrix} x+y & x^2+xy+y^2 \\ x-y & x^2-xy+y^2 \end{vmatrix}$.

11. Solve the simultaneous equations,

$$x - 2y = 3, \text{ and } 2x + 5y = 15.$$

SOLUTION. — Expressing the values of x and y in the notation of determinants, as in Article 3, we have

$$x = \frac{\begin{vmatrix} 3 & -2 \\ 15 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix}}, \text{ and } y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 15 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 2 & 5 \end{vmatrix}}.$$

Evaluating the above arrays by Article 2 gives

$$x = \tfrac{45}{9} = 5, \text{ and } y = \tfrac{9}{9} = 1.$$

12. Solve the simultaneous equations,

$$5x + 2y = 43, \quad x - \tfrac{1}{2}y = 5.$$

13. Solve the simultaneous equations,

$$2x + 3y = 22, \quad x - 2y = 4.$$

14. Solve the simultaneous equations,

$$8x + 7y = 208, \quad 5x = 60.$$

15. Solve the simultaneous equations,

$$\frac{1}{x} + \frac{1}{y} = 5, \quad \frac{2}{x} - \frac{3}{y} = 0.$$

16. Deduce the values of x' and y' in terms of x and y from the equations,

$$x = x' \cos\alpha - y' \sin\alpha, \quad y = x' \sin\alpha + y' \cos\alpha.$$

17. Express, in the notation of determinants, the condition that the two roots of the equation,

$$ax^2 + bx + c = 0,$$

shall be equal.

18. Express $a_1 : b_1 :: a_2 : b_2$ in determinant form.

19. Show that, if any element of a determinant of the second order is zero, the other element on the same diagonal may be replaced by any quantity whatever.

20. Show how any number may be expressed as a determinant of the second order.

4. Let us now solve the three simultaneous linear equations,

$$a_1x + b_1y + c_1z = \kappa_1,$$

$$a_2x + b_2y + c_2z = \kappa_2,$$

$$a_3x + b_3y + c_3z = \kappa_3.$$

Multiplying the first equation by $(b_2c_3 - b_3c_2)$, the second by $-(b_1c_3 - b_3c_1)$, the third by $(b_1c_2 - b_2c_1)$, and adding the resulting products, we readily obtain

$$x = \frac{\kappa_1b_2c_3 - \kappa_1b_3c_2 + \kappa_2b_3c_1 - \kappa_2b_1c_3 + \kappa_3b_1c_2 - \kappa_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1}. \quad . \ (1)$$

In a similar manner,

$$y = \frac{a_1\kappa_2c_3 - a_1\kappa_3c_2 + a_2\kappa_3c_1 - a_2\kappa_1c_3 + a_3\kappa_1c_2 - a_3\kappa_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1}, \quad . \ (2)$$

and

$$z = \frac{a_1b_2\kappa_3 - a_1b_3\kappa_2 + a_2b_3\kappa_1 - a_2b_1\kappa_3 + a_3b_1\kappa_2 - a_3b_2\kappa_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1}. \quad . \ (3)$$

These three fractions which express the values of x, y, and z have a common denominator which is a function of the coefficients of x, y, and z in the given simultaneous equations. This function,

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1, \quad . \ . \ . \ (4)$$

is called the determinant of the coefficients a, b_1, c_1, a_2, $\cdots$ a_3, $\cdots$ c_3, and in the notation of determinants it is commonly expressed by the symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \cdots\cdots \quad (4')$$

in which the coefficients are arranged in the same order as in the given equations.

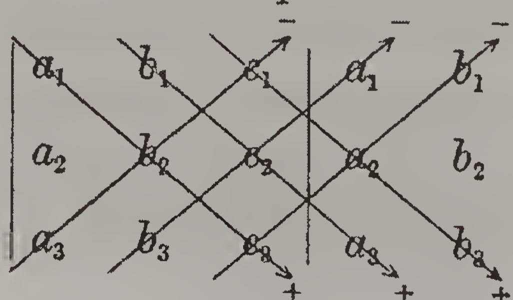
This symbol, like (3′) in Article 1, is called a determinant array, and a_1, b_1, c_1, a_2, ⋯ are called the elements or constituents of the determinant. The polynomial (4), like (3) in Article 1, is called the expansion of the determinant, and since each term of this expansion is the product of *three* elements, the determinant is said to be of the *third* order.

5. In the identical equation,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1,$$

the first member must be so interpreted that it shall represent the same function of a_1, b_1, c_1, a_2, ⋯ as the second member. The following is one method of so interpreting the first member:

Alongside the square array let the first two vertical ranks of elements be repeated in order; thus,



Now form the products of the elements lying in lines parallel to the diagonals of the original square, as shown above. The products formed from elements which lie on lines descending from left to right have the positive sign, the other products the negative sign.

By this method* *any determinant array of the third order* may be expanded and evaluated. In practice it will not be necessary to repeat the first and second vertical ranks of elements, but only to imagine them repeated.

6. The numerators of the fractions in Equations (1), (2), and (3) of Article 4 may also be written in the form of determinant arrays. Expressed in the notation of determinants, the values of x, y, and z in the given simultaneous equations are

$$x = \frac{\begin{vmatrix} \kappa_1 & b_1 & c_1 \\ \kappa_2 & b_2 & c_2 \\ \kappa_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & \kappa_1 & c_1 \\ a_2 & \kappa_2 & c_2 \\ a_3 & \kappa_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

* This device is due to Sarrus. See Scott's *Theory of Determinants*, page 10.

and
$$z = \frac{\begin{vmatrix} a_1 & b_1 & \kappa_1 \\ a_2 & b_2 & \kappa_2 \\ a_3 & b_3 & \kappa_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

NOTE.—As in Article 3, it may be observed that the numerator of the fraction expressing the value of x may be formed from the denominator of the same fraction by replacing a_1, a_2, and a_3, the coefficients of x, by the absolute terms κ_1, κ_2, and κ_3, respectively. Similarly for y and z.

EXAMPLES.

1. Expand $\begin{vmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$.

Ans. $x_0y_1 + x_2y_0 + x_1y_2 - x_2y_1 - x_0y_2 - x_1y_0$.

2. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$.

Ans. $1 + 8 + 27 - 6 - 6 - 6 = 18$.

3. Show that $\begin{vmatrix} 1 & x & y \\ 0 & \cos x & \sin y \\ 0 & \sin x & \cos y \end{vmatrix} = \cos(x + y)$.

4. Evaluate $\begin{vmatrix} 0 & l & m \\ -l & 0 & n \\ -m & -n & 0 \end{vmatrix}$. *Ans.* 0.

5. Evaluate $\begin{vmatrix} 3 & -3 & 4 \\ 3 & 2 & -2 \\ -1 & 2 & 1 \end{vmatrix}$.

6. Expand $\begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix}$.*

7. Show by expansion that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

8. Expand $\begin{vmatrix} L & n & m \\ n & M & l \\ m & l & N \end{vmatrix}$.

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* If three planes form a trihedral angle, the face angles of which are α, β, and γ, the above determinant is called the square of the sine of the solid angle in question, in analogy with

$$\begin{vmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{vmatrix} = \sin^2\alpha.$$

9. Show by expansion that

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = -m \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}.$$

10. Solve the simultaneous equations,

$$x - 2y + \tfrac{1}{2}z = -2,$$
$$3x - 2y = 0,$$
$$x + 2y + z = 12.$$

SOLUTION. — Expressing the values of x, y, and z in the notation of determinants, as in Article 6, we have

$$x = \frac{\begin{vmatrix} -2 & -2 & \frac{1}{2} \\ 0 & -2 & 0 \\ 12 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & \frac{1}{2} \\ 3 & -2 & 0 \\ 1 & 2 & 1 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 1 & -2 & \frac{1}{2} \\ 3 & 0 & 0 \\ 1 & 12 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & \frac{1}{2} \\ 3 & -2 & 0 \\ 1 & 2 & 1 \end{vmatrix}},$$

$$\text{and } z = \frac{\begin{vmatrix} 1 & -2 & -2 \\ 3 & -2 & 0 \\ 1 & 2 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & \frac{1}{2} \\ 3 & -2 & 0 \\ 1 & 2 & 1 \end{vmatrix}}.$$

Evaluating the above arrays by Article 5 gives

$$x = \tfrac{16}{8} = 2, \quad y = \tfrac{24}{8} = 3, \text{ and } z = \tfrac{32}{8} = 4.$$

11. Solve the simultaneous equations,

$$x + 2y - 3z = 13,$$
$$3x + y + 4z = 51,$$
$$\tfrac{1}{3}x + \tfrac{1}{4}y + \tfrac{1}{2}z = 7.$$

12. Solve the simultaneous equations,

$$x = 18 - 4y,\quad y = 71\tfrac{1}{2} - 7\tfrac{1}{2}z,\quad z = 10\tfrac{1}{3} - \tfrac{2}{3}x.$$

13. Solve the simultaneous equations,

$$\frac{x}{3} + \frac{y}{5} + \frac{z}{7} = 193,\quad \frac{x}{5} + \frac{y}{7} + \frac{z}{3} = 227,\quad \frac{x}{7} + \frac{y}{3} + \frac{z}{5} = 219.$$

14. Solve the simultaneous equations,

$$x + y + z = 29,\ x + 2y + 3z = 62,\ \tfrac{1}{2}x + \tfrac{1}{3}y + \tfrac{1}{4}z = 10.$$

15. Solve the simultaneous equations,

$$x + \tfrac{1}{2}y + \tfrac{1}{3}z = 32,\quad \tfrac{1}{3}x + \tfrac{1}{4}y + \tfrac{1}{5}z = 15,$$
$$\tfrac{1}{4}x + \tfrac{1}{5}y + \tfrac{1}{6}z = 12.$$

16. Solve the simultaneous equations,

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 6,\quad \frac{1}{x} + \frac{2}{y} + \frac{3}{z} = 14,\quad \frac{2}{x} + \frac{3}{y} + \frac{1}{z} = 11.$$

17. From the equations,

$$L\alpha = M\beta = N\gamma, \text{ and } a\alpha + b\beta + c\gamma = C,$$

deduce the values of α, β, and γ.

Ans. $\alpha = \dfrac{CMN}{aMN + bLN + cLM}$, etc.

18. Resolve $\dfrac{6x^2+22x+18}{x^3+6x^2+11x+6}$ into simpler fractions, using determinants wherever possible.

Ans. $\dfrac{1}{x+1}+\dfrac{2}{x+2}+\dfrac{3}{x+3}$.

7. The solution of the four simultaneous linear equations,

$$a_1x+b_1y+c_1z+d_1w=\kappa_1,$$
$$a_2x+b_2y+c_2z+d_2w=\kappa_2,$$
$$a_3x+b_3y+c_3z+d_3w=\kappa_3,$$
$$a_4x+b_4y+c_4z+d_4w=\kappa_4,$$

would show that the values of x, y, z, and w are expressed by fractions having a common denominator which is a function of the coefficients, a_1, b_1, c_1, d_1, a_2, $\cdots d_2$, $\cdots d_4$. This function is a determinant of the *fourth* order of which the above coefficients are the elements. The four-square array representing this determinant and the determinant in the expanded form are the members of the following identical equation:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \equiv \begin{aligned} &a_1b_2c_3d_4-a_1b_2c_4d_3-a_1b_3c_2d_4+a_1b_4c_2d_3\\ &+a_1b_3c_4d_2-a_1b_4c_3d_2-a_2b_1c_3d_4+a_2b_1c_4d_3\\ &+a_3b_1c_2d_4-a_4b_1c_2d_3-a_3b_1c_4d_2+a_4b_1c_3d_2\\ &+a_2b_3c_1d_4-a_2b_4c_1d_3-a_3b_2c_1d_4+a_4b_2c_1d_3\\ &+a_3b_4c_1d_2-a_4b_3c_1d_2-a_2b_3c_4d_1+a_2b_4c_3d_1\\ &+a_3b_2c_4d_1-a_4b_2c_3d_1-a_3b_4c_2d_1+a_4b_3c_2d_1. \end{aligned} \quad (1)$$

The solution of five simultaneous linear equations involving five unknown quantities would give rise to a determinant of the fifth order, which in its expanded form contains one hundred twenty (120) terms. The following is the array representing this determinant, the notation being the same as heretofore:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{vmatrix} \equiv a_1b_2c_3d_4e_5 \pm \text{etc.} \quad . \quad . \quad (2)$$

It is evident that the polynomials obtained in the above manner are far too cumbersome to be of any practical use in their expanded forms. The theory of determinants, however, enables us to manage not only such polynomials as the above, but also many other exceedingly complicated functions in a perfectly simple and easy manner. To quote Professor Sylvester, the theory of determinants is "an algebra upon an algebra; a calculus which enables us to combine and foretell the results of algebraical operations in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic." Aside from the power which determinants thus possess as instruments of analysis, the functions themselves have, by reason

of their great fertility, abundantly rewarded the careful study which has been bestowed upon them by the last two generations of mathematicians.

8. In the preceding articles it has appeared that the determinant of the *second* order involves *four* elements, the determinant of the *third* order *nine*, that of the *fourth* order *sixteen*, and that of the *fifth* order *twenty-five* elements.

In general, *the determinant of the* n*th order involves* n^2 *elements.*

Taking n letters,

$$a, b, c, \cdots h,$$

we may write the determinant array of the nth order thus:

$$\begin{vmatrix} a_1 & b_1 & c_1 \cdots h_1 \\ a_2 & b_2 & c_2 \cdots h_2 \\ a_3 & b_3 & c_3 \cdots h_3 \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n & b_n & c_n \cdots h_n \end{vmatrix} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

It is customary to speak of a *determinant array* simply as a *determinant.*

The horizontal ranks of elements are called **rows** of the determinant, and the vertical ranks are called **columns.** The rows are numbered from the top row

downward, and the columns from the left-hand column to the right.

In the above notation the number of the row to which a given element belongs is indicated by the subscript of the element, while the number of the column is indicated by the order of the letter in the alphabet.

Thus, in the determinant (1) of this article the element e_6 belongs to the sixth row and the fifth column.

In any determinant the diagonal from the upper left-hand corner to the lower right-hand corner is called the **principal diagonal**, and the other is called the **secondary diagonal**. The terms of the expansion which are the products of the elements on these diagonals, and it will appear in the sequence that there are such terms in the expansion of every determinant, are called respectively the **principal term** and the **secondary term**.

The element at the upper left-hand corner of the array is called the **leading element**, and the place which it occupies is called the **leading position**.

Thus, in the determinant (1), $a_1b_2c_3 \cdots h_n$ is the principal term, $a_nb_{n-1}c_{n-2} \cdots h_1$ is the secondary term, and a_1 is the leading element.

Another notation for the determinant array of the nth order is the following:

$$\begin{vmatrix} a_1' & a_1'' & a_1''' \cdots a_1^{(n)} \\ a_2' & a_2'' & a_2''' \cdots a_2^{(n)} \\ a_3' & a_3'' & a_3''' \cdots a_3^{(n)} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ a_n' & a_n'' & a_n''' \cdots a_n^{(n)} \end{vmatrix}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which the number of the *row* is indicated by the *subscript*, and the number of the *column* by the *superscript*. Thus, in the array just written, the element in the fifth row and the fourth column is a_5^{iv}.

The following is still another notation which is very much used:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1,n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2,n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3,n} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{n,n} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Here the number of the *row* is indicated by the *first* of the two subscripts, and the number of the *column* by the *second*. Thus, the element a_{35} of the above array is in the third row and the fifth column, while a_{53} is in the fifth row and the third column.

The elements of the array (3) should be read: "*a* one one," "*a* one two," "*a* one three," ..., "*a* one *n*," "*a* two one," "*a* two two," etc. In

this notation the a's are sometimes omitted, the double subscripts alone being used to represent the different elements. Thus:

$$\begin{vmatrix} 11 & 12 & 13 \ldots 1, n \\ 21 & 22 & 23 \ldots 2, n \\ 31 & 32 & 33 \ldots 3, n \\ & & \cdot \\ n1 & n2 & n3 \ldots n, n \end{vmatrix} \quad \text{. . . . .} \quad (3')$$

Instead of writing determinants in the form of square arrays a simpler method is often used *when it is perfectly well understood what the elements of the determinant are.* This method, named by Professor Sylvester the *umbral* method, consists simply in writing the principal term and enclosing it between the vertical bars. Thus, the determinants (1), (2), and (3) are respectively written, in this notation:

$$|\, a_1 b_2 c_3 \cdots h_n \,|, \quad \text{. . . . . . .} \quad (4)$$

$$|\, a_1' a_2'' a_3''' \cdots a_n^{(n)} \,|, \quad \text{. . . . . .} \quad (5)$$

and

$$|\, a_{11} a_{22} a_{33} \cdots a_{n,n} \,|. \quad \text{. . . . . .} \quad (6)$$

We may still further abridge (5) and (6) to

$$|\, a_1^{(n)} \,| \text{ and } |\, a_{1,n} \,|, \quad \text{. . . .} \quad (7), (8)$$

respectively.

CHAPTER II.

GENERAL DEFINITION OF A DETERMINANT.

In Articles 2 and 5 special rules were given for the interpretation of determinant arrays of the second and third orders, but no such simple rules can be given for the interpretation of arrays of higher orders. We now proceed, therefore, to develop a general definition of the functions known as determinants; that is, a general method of interpreting determinant arrays.

9. It is proved in elementary algebra that the number of permutations, or dispositions, of the members of a group of n things is

$$1 \cdot 2 \cdot 3 \cdots n, \text{ or } n!, \text{ or } \underline{|n}.$$

If the members of the group are the letters

$$a\ b\ c\ d\ e \cdots,$$

the order in which they have just been written is called the **natural order.** The natural order of the integers

$$1\ 2\ 3\ 4\ 5 \cdots$$

is, of course, the order of their magnitude, beginning with 1.

In one, and in only one, of the permutations of the members of a group, the members are arranged in their natural order. In every other permutation the natural order is more or less deranged.

Any two members of a group arranged in their natural order constitute a **permanence.** Thus, the pairs,

$$a\ b \qquad a\ c \qquad b\ c \qquad 1\ 2 \qquad 1\ 3 \qquad 2\ 3$$

are permanences.

Any two members of a group arranged in an order which is the reverse of the natural order constitute an **inversion.** Thus, the pairs,

$$b\ a \qquad c\ b \qquad d\ c \qquad 2\ 1 \qquad 3\ 2 \qquad 4\ 3$$

are inversions.

The number of permanences and the number of inversions in any permutation of the members of a group may be found by comparing each member of the group with each following member. Thus, in the permutation

$$a\ e\ d\ b\ c,$$

the permanences are

$$a\ e, \qquad a\ d, \qquad a\ b, \qquad a\ c, \qquad b\ c;$$

while the inversions are

$$e\ d, \qquad e\ b, \qquad e\ c, \qquad d\ b, \qquad d\ c.$$

In the permutation

$$3\ 4\ 2\ 5\ 1$$

there are four permanences and six inversions.*

The permutations of the members of a group are divided into two classes, the **even** or **positive** permutations and the **odd** or **negative** permutations.

Even permutations are those which contain an even number of inversions.

Odd permutations are those which contain an odd number of inversions.

The permutations

$$e\ a\ d\ b\ c, \qquad 3\ 4\ 2\ 5\ 1,$$

are even, because each contains an even number (six) of inversions; while the permutations

$$a\ e\ d\ b\ c, \qquad 3\ 4\ 2\ 1\ 5,$$

are odd, because each contains an odd number (five) of inversions.

The significance of the terms positive and negative, as applied to the classes of permutations, will appear later.

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* It may easily be shown that the sum of the number of permanences and the number of inversions in any permutation of n things is given by the formula,

$$\frac{n(n-1)}{2}.$$

10. THEOREM. — *If, in any permutation of the members of a group, two of the members be interchanged, the character of the permutation is changed, either from odd to even or from even to odd.*

We shall first consider the case in which the two members in question are adjacent. Let a and α be the two adjacent members, and represent collectively the members which precede $a\alpha$ by P and those which follow $a\alpha$ by R. The permutation may now be written

$$P\ a\ \alpha\ R.$$

Now, exchanging the order $a\alpha$ for the order αa does not in any way affect the relations of these two members to the members of either P or R. The only change in the number of inversions is due to the substitution of the order αa for the order $a\alpha$. If $a\alpha$ is a permanence, αa is an inversion, and the number of inversions is increased by one as the result of the exchange. If $a\alpha$ is an inversion, αa is a permanence, and the number of inversions is diminished by one. In either case, the interchange of a and α changes the number of inversions by one; hence, the character of the permutation is changed, either from even to odd or from odd to even.

If a and α be not adjacent, represent the mem-

bers included between them collectively by Q. The permutation may now be written

$$P\ a\ Q\ \alpha\ R.$$

Let q be the number of members in Q. We may bring the above permutation into the order

$$P\ a\ \alpha\ Q\ R$$

by interchanging α with each of the q members of Q in succession, beginning by interchanging it with the right-hand member. We have thus made q interchanges of adjacent members, and have accordingly changed the class of the permutation q times. The order $P\,a\,\alpha\,Q\,R$ may be changed to the order

$$P\ \alpha\ Q\ a\ R$$

by interchanging a with each member of (αQ) in succession, beginning by interchanging it with the left-hand member. This requires $(q+1)$ interchanges of adjacent members; that is, $(q+1)$ changes in the class of the permutation.

In the course of all the interchanges of adjacent members thus made in passing from the order $P\,a\,Q\,\alpha\,R$ to the order $P\,\alpha\,Q\,a\,R$ the class of the permutation has been changed

$$q+(q+1) \text{ or } (2q+1)$$

times. Since q is an integer, $(2q+1)$ is an odd

number, and the class of the permutation has been changed, either from odd to even or from even to odd. Hence the theorem.

11. THEOREM. — *Of all possible permutations of the members of a group, one-half are even and one-half are odd.*

Write down all the possible permutations. Now, let a new set of permutations be formed by fixing upon any two of the members and interchanging them in each permutation. The even permutations will thus be changed to odd, and the odd to even. That is, for every even permutation in the old set there is an odd one in the new, and *vice versa.* But, as is evident, the new set of permutations is the same as the old, only differently arranged. Hence, in either set there are as many even as there are odd permutations, or one-half the permutations are even and one-half are odd.

12. We are now prepared to give a general interpretation to the determinant array.

This array has already been written in the general form,

$$\begin{vmatrix} a_1' & a_1'' & a_1''' \cdots a_1^{(n)} \\ a_2' & a_2'' & a_2''' \cdots a_2^{(n)} \\ a_3' & a_3'' & a_3''' \cdots a_3^{(n)} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n' & a_n'' & a_n''' \cdots a_n^{(n)} \end{vmatrix}.$$

Write down all the products which can be formed by taking as factors one, and only one, element from each column and each row of the array. There are n! *such products;* for, since there must be in each product one element from each column, any one of the products without its subscripts may be written

$$a' \, a'' \, a''' \cdots a^{(n)},$$

and the n subscripts corresponding to the rows of the array may be appended to the members of the above group in $n!$ different ways, giving $n!$ different products.

Of these products, one half involve the even permutations, and the other half the odd permutations of the subscripts 1, 2, 3, ⋯ n (Art. 11). *Now give to those products which involve the* EVEN *permutations of the subscripts the* POSITIVE *sign, and to those which involve the* ODD *permutations the* NEGATIVE *sign, and take their algebraic sum. The result is the expanded form of the determinant.*

[The portions of this article printed in italics read consecutively, and furnish a general rule for the expansion of determinant arrays. The reader will do well to verify the rule by applying it to the determinants in Articles 1 to 7. It should be mentioned that the method of expansion just given is of little practical value. It constitutes a general definition of a determinant, however, and as such will be used as a basis for the theorems relating to the prop-

erties of determinants given in the next chapter. For this reason it must be thoroughly mastered.]

13. Let us assume any term of the expansion of the determinant written in the preceding article, as

$$\pm a_h{}' a_i{}'' a_j{}''' \cdots a_l{}^{(n)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $hij \cdots l$ is any permutation of the subscripts $1, 2, 3, \cdots n$, involving any number, μ (say), of inversions.

Interchange any two of the elements the subscripts of which give rise to one of the μ inversions, and continue this process till all the inversions in the arrangement of the subscripts have disappeared. This will require a number, m (say), of interchanges of elements, m being even or odd according as μ is even or odd, and as a final result the elements will be so arranged that their subscripts are in the natural order, $1\,2\,3 \cdots n$. The term under consideration may now be written

$$\pm a_1{}^{(p)} a_2{}^{(q)} a_3{}^{(r)} \cdots a_n{}^{(t)}, \quad . \quad . \quad . \quad . \quad (2)$$

in which $pqr \cdots t$ is a certain permutation of the superscripts $', '', ''', \cdots {}^{(n)}$.

Now, each of the m interchanges of two elements, by which the second form of the term has been obtained, has changed the class of the permutation of the superscripts from even to odd, or *vice versa.*

The original permutation was even, the superscripts having been arranged in the natural order.*

Hence, the permutation $pqr \cdots t$ is even or odd according as m and μ are even or odd. The two permutations, $hij \cdots l$ and $pqr \cdots t$ are, therefore, of the same class, and the term under consideration will have the same sign, whether its sign be determined by its subscripts when in the form (1), or by its superscripts when in the form (2).

It follows that it is immaterial whether we write the terms of the expansion of an array so that the *superscripts* of the elements are arranged in the natural order and consider each term as positive or negative according as the permutation of its *subscripts* is even or odd, or write the terms so that the *subscripts* are arranged in the natural order, and consider each term as positive or negative according as the permutation of its *superscripts* is even or odd.

NOTE. — It may also be shown that if we write down all the products which can be formed by taking as factors one element from each column and each row of the array, the factors in each product being arranged neither with reference to the order of the subscripts nor of the superscripts, and take each product positive or negative, according as the sum of the number of inversions in the arrangement of the subscripts and the number of inversions in the arrangement of the superscripts is even

* The number of inversions in the natural order is zero, an even number.

or odd, the algebraic sum of the resulting terms will be the expanded form of the determinant.

This will furnish a good exercise for the student.

14. The determinants (2) and (3) of Article 8 may be written in the simpler forms

$$\Sigma \pm (a_h{}' a_i{}'' a_j{}''' \cdots a_l{}^{(n)}), \text{ or } \Sigma \pm (a_1{}^{(p)} a_2{}^{(q)} a_3{}^{(r)} \cdots a_n{}^{(t)}), \quad (1)$$

and

$$\Sigma \pm (a_{h,1} a_{i,2} a_{j,3} \cdots a_{l,n}), \text{ or } \Sigma \pm (a_{1,p} a_{2,q} a_{3,r} \cdots a_{n,t}), \quad (2)$$

respectively; a notation which is suggested by the principles explained in the two preceding articles.

CHAPTER III.

PROPERTIES OF DETERMINANTS.

From the definition of a determinant contained in Articles 12 and 13 of the last chapter, we now proceed to derive the more important theorems relating to the properties of determinants.

and the column into the row, etc.

15. THEOREM. — *The value of a determinant is not changed by changing the columns into corresponding rows, and the rows into corresponding columns;* that is,

$$\begin{vmatrix} a_1{}'a_1{}'' \cdots a_1{}^{(n)} \\ a_2{}'a_2{}'' \cdots a_2{}^{(n)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n{}'a_n{}'' \cdots a_n{}^{(n)} \end{vmatrix} = \begin{vmatrix} a_1{}' & a_2{}' & \cdots & a_n{}' \\ a_1{}'' & a_2{}'' & \cdots & a_n{}'' \\ \cdot & \cdot & \cdot & \cdot \\ a_1{}^{(n)} & a_2{}^{(n)} & \cdots & a_n{}^{(n)} \end{vmatrix}.$$

A determinant may be expanded either by permuting the subscripts while the superscripts remain in the natural order, or by permuting the superscripts while the subscripts remain in the natural order (Articles 12 and 13). Now, it is evident, since the subscripts refer to the rows and the superscripts to the columns of the array, or *vice versa*, that changing from one of these methods of expan-

sion to the other amounts to the same thing as changing the columns of the array into corresponding rows, and the rows into corresponding columns. Hence the theorem.

It follows that whatever theorem is demonstrated in regard to the rows of a determinant is also true in regard to the columns, and *vice versa*.

16. THEOREM. — *Interchanging any two rows (or columns) of a determinant changes the sign of the determinant;* thus,

$$\begin{vmatrix} a_1' a_1'' a_1''' \cdots a_1^{(n)} \\ a_2' a_2'' a_2''' \cdots a_2^{(n)} \\ a_3' a_3'' a_3''' \cdots a_3^{(n)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n' a_n'' a_n''' \cdots a_n^{(n)} \end{vmatrix} = - \begin{vmatrix} a_3' a_3'' a_3''' \cdots a_3^{(n)} \\ a_2' a_2'' a_2''' \cdots a_2^{(n)} \\ a_1' a_1'' a_1''' \cdots a_1^{(n)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n' a_n'' a_n''' \cdots a_n^{(n)} \end{vmatrix} = \begin{vmatrix} a_3' a_3''' a_3'' \cdots a_3^{(n)} \\ a_2' a_2''' a_2'' \cdots a_2^{(n)} \\ a_1' a_1''' a_1'' \cdots a_1^{(n)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n' a_n''' a_n'' \cdots a_n^{(n)} \end{vmatrix}.$$

Represent the given determinant by Δ, and the determinant obtained by interchanging any two rows of Δ by Δ'. Let

$$\pm\, a_h' a_i'' a_j''' a_k^{\text{iv}} \cdots a_m^{(n)}$$

be any term of Δ. If Δ' be formed from Δ by interchanging the rows whose subscripts are i and k (say), the corresponding term of Δ' will be

$$\pm\, a_h' a_k'' a_j''' a_i^{\text{iv}} \cdots a_m^{(n)}.$$

This is evidently one of the terms of Δ, if we disregard its sign (Art. 12). Since, however,

$h\,i\,j\,k \cdots m$ and $h\,k\,j\,i \cdots m$ are not permutations of the same class, the sign of this term in Δ must be $\mp$ instead $\pm$.

Hence, for every term

$$\pm a_h{}' a_k{}'' a_j{}''' a_i{}^{\text{iv}} \cdots a_m{}^{(n)}$$

of Δ', there is a term

$$\mp a_h{}' a_k{}'' a_j{}''' a_i{}^{\text{iv}} \cdots a_m{}^{(n)}$$

in Δ. Therefore $\Delta = -\Delta'$, which is the theorem.

17. An element in the kth row and the sth column may be brought into the leading position by $(k-1)$ interchanges of adjacent rows and $(s-1)$ interchanges of adjacent columns. Since, by the preceding article, each such interchange changes the sign of the determinant, the resulting determinant will be *positive* or *negative* according as $(k+s-2)$, the total number of interchanges, is even or odd; that is, according as $(k+s)$ is *even* or *odd*.

18. THEOREM.—*If two rows (or columns) of a determinant are identical, the determinant is equal to zero;* thus,

$$\Delta \equiv \begin{vmatrix} a_1{}' & a_1{}'' \cdots a_1{}^{(n)} \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ a_k{}' & a_k{}'' \cdots a_k{}^{(n)} \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ a_k{}' & a_k{}'' \cdots a_k{}^{(n)} \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ a_n{}' & a_n{}'' \cdots a_n{}^{(n)} \end{vmatrix} = 0.$$

For, by interchanging the two identical rows (Art. 16), we obtain

$$\Delta = -\Delta,$$

whence $$\Delta = 0.$$

19. THEOREM.—*If each element of any column (or row) of a determinant is the sum of two or more quantities, the determinant can be expressed as the sum of two or more determinants;* thus,

$$\begin{vmatrix} (a_1' + b_1' + c_1' + \cdots) & a_1'' \cdots a_1^{(n)} \\ (a_2' + b_2' + c_2' + \cdots) & a_2'' \cdots a_2^{(n)} \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ (a_n' + b_n' + c_n' + \cdots) & a_n'' \cdots a_n^{(n)} \end{vmatrix} = \begin{vmatrix} a_1' \, a_1'' \cdots a_1^{(n)} \\ a_2' \, a_2'' \cdots a_2^{(n)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n' \, a_n'' \cdots a_n^{(n)} \end{vmatrix}$$

$$+ \begin{vmatrix} b_1' \, a_1'' \cdots a_1^{(n)} \\ b_2' \, a_2'' \cdots a_2^{(n)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ b_n' \, a_n'' \cdots a_n^{(n)} \end{vmatrix} + \begin{vmatrix} c_1' \, a_1'' \cdots a_1^{(n)} \\ c_2' \, a_2'' \cdots a_2^{(n)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ c_n' \, a_n'' \cdots a_n^{(n)} \end{vmatrix} + \cdots$$

In the notation explained in Article 14, the given determinant may be written

$$\Sigma \pm [(a_h' + b_h' + c_h' + \cdots) \; a_i'' \, a_j''' \cdots a_l^{(n)}].$$

This may be resolved into the following:

$$\Sigma \pm (a_h' a_i'' a_j''' \cdots a_l^{(n)}) + \Sigma \pm (b_h' a_i'' a_j''' \cdots a_l^{(n)})$$
$$+ \Sigma \pm (c_h' a_i'' a_j''' \cdots a_l^{(n)}) + \cdots$$

These terms represent, in the notation of Article 14, the determinants in the second member of the above equation, respectively. Hence the theorem.

20. THEOREM. — *Multiplying each element of a column (or row) of a determinant by a given factor multiplies the determinant by that factor;* thus,

$$\begin{vmatrix} ma_1' & a_1'' \cdots a_1^{(n)} \\ ma_2' & a_2'' \cdots a_2^{(n)} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ ma_n' & a_n'' \cdots a_n^{(n)} \end{vmatrix} = m \begin{vmatrix} a_1' & a_1'' \cdots a_1^{(n)} \\ a_2' & a_2'' \cdots a_2^{(n)} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ a_n' & a_n'' \cdots a_n^{(n)} \end{vmatrix}.$$

Since each term of the expansion contains one, and only one, element from the column in question, if each element of this column be multiplied by a given factor, each term of the expansion will be so multiplied; that is, the determinant will be multiplied by the given factor, which is the theorem.

It follows that

If each element of a column (or row) of a determinant is zero, the determinant vanishes.

21. THEOREM. — *If the elements of any row (or column) of a determinant have a common ratio to the corresponding elements of any other row (or column), the determinant is equal to zero;* thus,

$$\begin{vmatrix} a_1' & a_1'' \cdots & a_1^{(n)} \\ a_2' & a_2'' \cdots & a_2^{(n)} \\ ma_2' & ma_2'' \cdots & ma_2^{(n)} \\ \cdot & \cdot \quad \cdot & \cdot \\ a_n' & a_n'' \cdots & a_n^{(n)} \end{vmatrix} = 0.$$

The common ratio (m) may be written outside the array as a factor of the whole determinant (Art. 20). The determinant then vanishes, because it has two identical rows (Art. 18).

22. THEOREM. — *If each element of a column (or row) of a determinant be multiplied by a given factor, and the product added to the corresponding element of any other column (or row), the value of the determinant will not be changed;* thus,

$$\begin{vmatrix} a_1' & a_1'' & a_1''' \cdots a_1^{(n)} \\ a_2' & a_2'' & a_2''' \cdots a_2^{(n)} \\ \cdot & \cdot & \cdot \quad \cdot \\ a_n' & a_n'' & a_n''' \cdots a_n^{(n)} \end{vmatrix} = \begin{vmatrix} a_1' & (a_1''+ma_1''') & a_1''' \cdots a_1^{(n)} \\ a_2' & (a_2''+ma_2''') & a_2''' \cdots a_2^{(n)} \\ \cdot & \cdot & \cdot \quad \cdot \\ a_n' & (a_n''+ma_n''') & a_n''' \cdots a_n^{(n)} \end{vmatrix}.$$

The second member of this equation may be written (Art. 19),

$$\begin{vmatrix} a_1' & a_1'' & a_1''' \cdots a_1^{(n)} \\ a_2' & a_2'' & a_2''' \cdots a_2^{(n)} \\ \cdot & \cdot & \cdot \quad \cdot \\ a_n' & a_n'' & a_n''' \cdots a_n^{(n)} \end{vmatrix} + \begin{vmatrix} a_1' & ma_1''' & a_1''' \cdots a_1^{(n)} \\ a_2' & ma_2''' & a_2''' \cdots a_2^{(n)} \\ \cdot & \cdot & \cdot \quad \cdot \\ a_n' & ma_n''' & a_n''' \cdots a_n^{(n)} \end{vmatrix}.$$

The second of these determinants vanishes by the theorem demonstrated in the preceding article, and the given equation thus reduces to an identity. Hence the theorem.

It may likewise be shown that we may combine, in the same manner as above, any number of columns (or rows) without changing the value of the determinant. Care must be taken, however, not to modify in any way the elements to which we add multiples of the corresponding elements from other columns (or rows). The above theorem in its general form is illustrated by the following equation:

$$\begin{vmatrix} a_1' & a_1'' & a_1''' & a_1^{\text{iv}} \cdots a_1^{(n)} \\ a_2' & a_2'' & a_2''' & a_2^{\text{iv}} \cdots a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ a_n' & a_n'' & a_n''' & a_n^{\text{iv}} \cdots a_n^{(n)} \end{vmatrix} =$$

$$\begin{vmatrix} (a_1' + la_1'' + ma_1''' + \cdots) & a_1'' & a_1''' & a_1^{\text{iv}} \cdots a_1^{(n)} \\ (a_2' + la_2'' + ma_2''' + \cdots) & a_2'' & a_2''' & a_2^{\text{iv}} \cdots a_2^{(n)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ (a_n' + la_n'' + ma_n''' + \cdots) & a_n'' & a_n''' & a_n^{\text{iv}} \cdots a_n^{(n)} \end{vmatrix}.$$

23. THEOREM.—*If a determinant is a rational integral function of* a, *and also a rational integral function of* b, *such that, if* b *is substituted for* a, *the determinant vanishes, then is* (a−b) *a factor of the determinant.* For example, the determinant

$$\begin{vmatrix} a-m & na & a^2 \\ b-m & nb & b^3 \\ p & q & r \end{vmatrix}$$

contains the factor $(a-b)$.

The expansion of the determinant may be written in the form

$$\Delta = \lambda_0 + \lambda_1 a + \lambda_2 a^2 + \lambda_3 a^3 + \cdots, \quad . \quad . \quad . \quad . \quad (1)$$

in which λ_0, λ_1, λ_2, $\cdots$ are independent of a. Since upon substituting b for a the determinant vanishes, we have

$$0 = \lambda_0 + \lambda_1 b + \lambda_2 b^2 + \lambda_3 b^3 + \cdots \quad . \quad . \quad . \quad . \quad (2)$$

Subtracting (2) from (1) gives

$$\Delta = \lambda_1(a-b) + \lambda_2(a^2 - b^2) + \lambda_3(a^3 - b^3) + \cdots,$$

which is divisible by $(a-b)$. Hence the theorem.

The above theorem is often useful in factoring determinants without previously expanding them. Thus, the determinant

$$\Delta \equiv \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

vanishes when $a=b$, when $a=c$, and when $b=c$, and therefore contains the factors $(a-b)$, $(a-c)$, and $(b-c)$. Since the product of these three factors is of the same degree as Δ, these are readily seen to be the only factors. Hence

$$\Delta = (b-c)(a-c)(a-b).$$

EXAMPLES.

1. Tell the signs of the terms

$$a_5b_4c_3d_2e_1,\quad a_3e_2c_4d_1b_5,\quad b_2a_1e_3c_4d_5,\quad b_3c_2a_1d_5e_4,$$

$$a_4b_3c_2e_1d_5,\quad b_1a_2d_3c_5e_4,\quad e_5d_4c_3b_2a_1,\ \text{and}\ e_1a_3d_4b_5c_2$$

of the determinant $|\, a_1b_2c_3d_4e_5 \,|$.

2. Expand the determinant

$$\begin{vmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{vmatrix}.$$

Prove the following equalities without expansion:

3. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix}.$$

4. $$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} c_1 & b_1 & a_1 \\ c_3 & b_3 & a_3 \\ c_2 & b_2 & a_2 \end{vmatrix}.$$

5. $$\begin{vmatrix} 2 & 1 & 2 \\ 3 & 3 & 1 \\ 1 & 5 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 6 \\ 1 & 6 & 3 \\ 1 & 10 & 3 \end{vmatrix} = \begin{vmatrix} 20 & 1 & 2 \\ 9 & 3 & 1 \\ 7 & 5 & 1 \end{vmatrix}.$$

6. $$\begin{vmatrix} 6 & 1 & -7 \\ 5 & -10 & 5 \\ 4 & 3 & -7 \end{vmatrix} = 0.$$

7. $\begin{vmatrix} -a+b+c & a & -b \\ a-b+c & b & -c \\ a+b-c & c & -a \end{vmatrix} = \begin{vmatrix} c & a & -b \\ a & b & -c \\ b & c & -a \end{vmatrix}.$

8. $\begin{vmatrix} 0 & am & -abn \\ -l & 0 & bn \\ l & -m & 0 \end{vmatrix} = 0.$

9. Rewrite the determinant in Ex. 2 so that the element n shall occupy the leading position.

Find the values of x which satisfy the following equations:

10. $\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = 0.$

11. $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0.$

12. $\begin{vmatrix} 0 & 1 & \sin x \\ \cos x & 0 & \sin x \\ \cos x & 1 & 0 \end{vmatrix} = 1.$ *Ans.* $x = \frac{\pi}{4}.$

13. Prove that

$$\begin{vmatrix} 1 & x-a & y-b \\ 1 & x_1-a & y_1-b \\ 1 & x_2-a & y_2-b \end{vmatrix} = \begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = \begin{vmatrix} 1 & x & y \\ 0 & x_1-x & y_1-y \\ 0 & x_2-x & y_2-y \end{vmatrix}.$$

14. If $\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = 0,$

prove that $$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1).$$

Resolve the following determinants into factors:

15. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$.

16. $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ac \\ c & c^2 & ab \end{vmatrix}$.

17. $\begin{vmatrix} 1 & bc & a^3 \\ 1 & ac & b^3 \\ 1 & ab & c^3 \end{vmatrix}$.

18. $\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$.

19. $\begin{vmatrix} 1 & a & a^2 & a^4 \\ 1 & b & b^2 & b^4 \\ 1 & c & c^2 & c^4 \\ 1 & d & d^2 & d^4 \end{vmatrix}$.

Ans. $(a+b+c+d)(a-b)$
$(a-c)(a-d)(b-c)$
$(b-d)(c-d)$.

20. $\begin{vmatrix} a & b+c & a^2 \\ b & a+c & b^2 \\ c & a+b & c^2 \end{vmatrix}$.

21. $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

22. Resolve the determinant

$$\Delta \equiv \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}$$

into linear factors.

SOLUTION. — Let us designate by L_k the kth row and by $L^{(s)}$ the sth column of the given determinant. Then, dividing L_2 by bc, L_3 by ac, and L_4 by ab, we have

$$\Delta = a^2b^2c^2 \begin{vmatrix} 0 & 1 & 1 & 1 \\ \frac{1}{bc} & 0 & \frac{c}{b} & \frac{b}{c} \\ \frac{1}{ac} & \frac{c}{a} & 0 & \frac{a}{c} \\ \frac{1}{ab} & \frac{b}{a} & \frac{a}{b} & 0 \end{vmatrix}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Multiplying L' by abc, L'' by a, L''' by b, and L^{iv} by c, this becomes

$$\Delta = \begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This may now be put in the form

$$\Delta = \begin{vmatrix} a+b+c & a & b & c \\ a+b+c & 0 & c & b \\ a+b+c & c & 0 & a \\ a+b+c & b & a & 0 \end{vmatrix},$$

by taking for the first column

$$L' + L'' + L''' + L^{iv} \text{ (Art. 22)},$$

which shows that the determinant contains the factor $(a + b + c)$.

In the same manner, by taking for the first column the sums

$L' + L'' - L''' - L^{iv}$, $L' - L'' + L''' - L^{iv}$, and $L' - L'' - L''' + L^{iv}$,

it may be shown that the determinant (2) is divisible by

$$(-a+b+c),\ (a-b+c),\ \text{and}\ (a+b-c),$$

respectively.

The product of the four factors thus found being, like (2), of the fourth degree in a, b, and c, we infer that (2) is of the form

$$\Delta = \lambda(a+b+c)(-a+b+c)(a-b+c)(a+b-c), \quad (3)$$

in which λ represents a factor independent of a, b, and c. This factor may be found by comparing any term of the expansion of (2) with the corresponding term of the expansion of (3), say the term containing a^4. The term a^4 in the expansion of (2) has the same sign as the term $a_2b_1c_4d_3$ of the determinant (1), Article 7, which is positive (Art. 12). The corresponding term in the expansion of (3) is $-\lambda a^4$. Whence, $\lambda = -1$, and we have

$$\Delta = -(a+b+c)(-a+b+c)(a-b+c)(a+b-c).$$

It appears that $-\Delta$ represents sixteen times the square of the area of the triangle having the sides a, b, and c.

23. Show that the determinant

$$\Delta \equiv \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$$

is equal to the product

$$(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d).$$

NOTE.—It may be shown that $-\Delta$ represents sixteen times the square of the area of the quadrilateral inscribed in a circle and having for sides $-a$, b, c, and d.

24. By applying the theorems demonstrated in this chapter to the determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

deduce the common theorems in proportion.

CHAPTER IV.

DETERMINANT MINORS.

To expand or evaluate determinants of higher than the second or third order, they are resolved into determinants of lower orders. The method by which this is accomplished is explained in the articles immediately following.

24. In order to find the terms containing the element a_1' of the determinant of the nth order,

$$\Delta \equiv \begin{vmatrix} a_1' & a_1'' & a_1''' \cdots a_1^{(n)} \\ a_2' & a_2'' & a_2''' \cdots a_2^{(n)} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ a_n' & a_n'' & a_n''' \cdots a_n^{(n)} \end{vmatrix}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

let each element of the first row except a_1' be zero; the determinant thus becomes

$$\begin{vmatrix} a_1' & 0 & 0 \quad \cdots 0 \\ a_2' & a_2'' & a_2''' \cdots a_2^{(n)} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ a_n' & a_n'' & a_n''' \cdots a_n^{(n)} \end{vmatrix}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Each term of the expansion of this array contains as a factor one, and only one, element from the first row (Art. 12), and the only terms which do not

reduce to zero are those formed by taking a_1' as the factor from the first row. Hence, the expansion of (2) contains the terms of the expansion of Δ which involve a_1', and it contains no other terms.

Also, those terms of the expansion of (2) which do not vanish, can contain no other element than a_1' from the first *column*. Each term is, therefore, formed by multiplying by a_1' some one of the $(n-1)!$ products of $(n-1)$ elements formed by taking one element from each column and row of

$$\begin{array}{lll} a_2'' & a_2''' \cdots & a_2^{(n)}. \\ a_3'' & a_3''' \cdots & a_3^{(n)} \\ . & . \quad . & . \quad . \\ a_n'' & a_n''' \cdots & a_n^{(n)} \end{array}$$

Any one of these products without the subscripts may be written

$$a'' \quad a''' \cdots a^{(n)},$$

and the different products may be formed by distributing the $(n-1)$ subscripts

$$2, 3, \cdots n$$

in every one of the $(n-1)!$ possible ways. The sign of the term formed by multiplying any one of these products by a_1' is determined by the class of the permutation of the subscripts

$$1, 2, 3, \cdots n,$$

and, since the subscript 1 remains in the first position, this permutation is of the same class as that of the subscripts

$$2,\ 3, \cdots n.$$

It follows, therefore, that the algebraic sum of all the products which are multiplied by a_1' is the determinant

$$\begin{vmatrix} a_2'' & a_2''' \cdots a_2^{(n)} \\ a_3'' & a_3''' \cdots a_3^{(n)} \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'' & a_n''' \cdots a_n^{(n)} \end{vmatrix} . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence,

$$\begin{vmatrix} a_1' & 0 & 0 & \cdots 0 \\ a_2' & a_2'' & a_2''' & \cdots a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ a_n' & a_n'' & a_n''' & \cdots a_n^{(n)} \end{vmatrix} = a_1' \begin{vmatrix} a_2'' & a_2''' \cdots a_2^{(n)} \\ a_3'' & a_3''' \cdots a_3^{(n)} \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'' & a_n''' \cdots a_n^{(n)} \end{vmatrix} . \quad . \quad (4)$$

The determinant (3) is called the **co-factor** of the element a_1' in the determinant Δ (Eq. 1). It may be formed from Δ by deleting the row and column to which the element a_1' belongs; that is, the first row and the first column.

25. The co-factor of any element, $a_k^{(s)}$, of a determinant may be found by bringing that element into the leading position (Art. 17), and then deleting the first row and the first column. The co-factor thus obtained will be positive or negative

according as $(k+s)$ is even or odd. But the process of bringing $a_k^{(s)}$ into the leading position does not in any way change the relations of the elements in the remaining rows and columns; hence

To find the co-factor of any element, $a_k^{(s)}$, *of a determinant, delete the row and the column to which the element belongs, and give the resulting determinant the positive sign when* $(k+s)$ *is even, and the negative sign when* $(k+s)$ *is odd.*

The co-factor of any element, $a^{(s)}$, of a determinant is represented by the symbol

$$A_k^{(s)}.$$

The sign of this co-factor is

$$(-1)^{k+s},$$

but the expression $A_k^{(s)}$ is generally considered as including the sign within itself, and is accordingly written as positive.

The co-factors of the various elements of the determinant

$$\begin{vmatrix} a_1' & a_1'' & a_1''' \\ a_2' & a_2'' & a_2''' \\ a_3' & a_3'' & a_3''' \end{vmatrix}$$

are as follows:

$$A_1' = \begin{vmatrix} a_2'' & a_2''' \\ a_3'' & a_3''' \end{vmatrix}, \quad A_1'' = -\begin{vmatrix} a_2' & a_2''' \\ a_3' & a_3''' \end{vmatrix}, \quad A_1''' = \begin{vmatrix} a_2' & a_2'' \\ a_3' & a_3'' \end{vmatrix},$$

$$A_2' = -\begin{vmatrix} a_1'' & a_1''' \\ a_3'' & a_3''' \end{vmatrix}, \quad A_2'' = \begin{vmatrix} a_1' & a_1''' \\ a_3' & a_3''' \end{vmatrix}, \quad A_2''' = -\begin{vmatrix} a_1' & a_1'' \\ a_3' & a_3'' \end{vmatrix},$$

$$A_3' = \begin{vmatrix} a_1'' & a_1''' \\ a_2'' & a_2''' \end{vmatrix}, \quad A_3'' = -\begin{vmatrix} a_1' & a_1''' \\ a_2' & a_2''' \end{vmatrix}, \quad A_3''' = \begin{vmatrix} a_1' & a_1'' \\ a_2' & a_2'' \end{vmatrix}.$$

26. The result obtained by deleting the row and column to which any element of a determinant belongs is called the **minor** of the determinant with respect to that element; thus the minor with respect to the element $a_k^{(s)}$ is obtained by deleting the kth row and the sth column.

The *co-factor* of $a_k^{(s)}$ in the same determinant is (Art. 25) equal to this *minor* multiplied by the sign-factor $(-1)^{k+s}$.

The minor of the determinant Δ with respect to the element $a_k^{(s)}$ is represented by the symbol

$$\Delta_{(k}^{(s}.$$

Accordingly, the co-factor $A_k^{(s)}$ of the same element may be written

$$(-1)^{k+s}\Delta_{(k}^{(s}.$$

27. The algebraic sum of the $(n-1)!$ terms of Δ which contain the element $a_k^{(s)}$ is

$$a_k^{(s)} A_k^{(s)}.$$

The algebraic sums of all the terms which contain the successive elements

$$a_k{}', \; a_k{}'', \; a_k{}''', \; \cdots, \; a_k{}^{(n)}$$

of the kth row, are respectively

$$a_k{}'A_k{}', \; a_k{}''A_k{}'', \; a_k{}'''A_k{}''', \; \cdots, \; a_k{}^{(n)}A_k{}^{(n)},$$

n in number. Each one of these sums is composed of $(n-1)!$ of the terms of the determinant Δ, and no one of these terms is found in any other sum. There are then in all of them $n \times (n-1)!$, or $n!$ terms, no two alike. They are the $n!$ terms of the expansion of Δ. Hence

$$\Delta = a_k{}'A_k{}' + a_k{}''A_k{}'' + a_k{}'''A_k{}''' + \cdots + a_k{}^{(n)}A_k{}^{(n)}. \qquad (1)$$

In a similar manner it may be shown that

$$\Delta = a_1{}^{(s)}A_1{}^{(s)} + a_2{}^{(s)}A_2{}^{(s)} + a_3{}^{(s)}A_3{}^{(s)} + \cdots + a_n{}^{(s)}A_n{}^{(s)}. \quad (2)$$

By means of Formulæ (1) *and* (2) *we may express any determinant in terms of determinants of one lower order;* thus, applying Formula (1) to the determinant

$$\begin{vmatrix} a_1{}' & a_1{}'' & a_1{}''' \\ a_2{}' & a_2{}'' & a_2{}''' \\ a_3{}' & a_3{}'' & a_3{}''' \end{vmatrix}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

gives, making $k=1$,

$$\Delta = a_1' \begin{vmatrix} a_2'' & a_2''' \\ a_3'' & a_3''' \end{vmatrix} - a_1'' \begin{vmatrix} a_2' & a_2''' \\ a_3' & a_3''' \end{vmatrix} + a_1''' \begin{vmatrix} a_2' & a_2'' \\ a_3' & a_3'' \end{vmatrix};$$

making $k=2$,

$$\Delta = -a_2' \begin{vmatrix} a_1'' & a_1''' \\ a_3'' & a_3''' \end{vmatrix} + a_2'' \begin{vmatrix} a_1' & a_1''' \\ a_3' & a_3''' \end{vmatrix} - a_2''' \begin{vmatrix} a_1' & a_1'' \\ a_3' & a_3'' \end{vmatrix};$$

making $k=3$,

$$\Delta = a_3' \begin{vmatrix} a_1'' & a_1''' \\ a_2'' & a_2''' \end{vmatrix} - a_3'' \begin{vmatrix} a_1' & a_1''' \\ a_2' & a_2''' \end{vmatrix} + a_3''' \begin{vmatrix} a_1' & a_1'' \\ a_2' & a_2'' \end{vmatrix}.$$

Formula (2) may be applied in a similar manner.

Since, in Formulæ (1) and (2), the co-factors

$$A_k', A_k'', \cdots, A_k^{(n)} \quad \text{or} \quad A_1^{(s)}, A_2^{(s)}, \cdots, A_n^{(s)},$$

are themselves determinants, they may be resolved into determinants of still lower order in the same manner.

By this process any determinant may be expressed in terms of determinants of the third or second order, convenient rules for the expansion of which have already been given.

28. In the determinant Δ of the preceding article, if the row

$$a_k' \quad a_k'' \quad \cdots \quad a_k^{(n)}$$

be identical with the row

$$a_h' \quad a_h'' \quad \cdots \quad a_h^{(n)},$$

then, instead of writing the expansion as in Equation (1), it may be written

$$a_h'A_k' + a_h''A_k'' + \cdots + a_h^{(n)}A_k^{(n)}.$$

The above expression is, then, a form for the expansion of a determinant in which the hth and kth rows are identical. Hence (Art. 18), *when* h *and* k *are different subscripts,*

$$a_h'A_k' + a_h''A_k'' + a_h'''A_k''' + \cdots + a_h^{(n)}A_k^{(n)} = 0. \quad . \quad . \quad (1)$$

Similarly, *when* p *and* s *are different superscripts,*

$$a_1^{(p)}A_1^{(s)} + a_2^{(p)}A_2^{(s)} + a_3^{(p)}A_3^{(s)} + \cdots + a_n^{(p)}A_n^{(s)} = 0. \quad . \quad (2)$$

Thus, in reference to the determinant (3) of the preceding article,

$$a_2'A_3' + a_2''A_3'' + a_2'''A_3''' = \begin{vmatrix} a_2' & a_2'' & a_2''' \\ a_1' & a_1'' & a_1''' \\ a_2' & a_2'' & a_2''' \end{vmatrix} = 0.$$

EXAMPLES.

1. Expand the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

SOLUTION.—Applying Equation (1) of Article 27, letting $k = 1$, gives

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1 \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

Each of the above determinants may be resolved in the same manner, giving

$$\begin{aligned} \Delta = a_1b_2 \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} - a_1c_2 \begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix} + a_1d_2 \begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix} \\ - a_2b_1 \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} + b_1c_2 \begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix} - b_1d_2 \begin{vmatrix} a_3 & c_3 \\ a_4 & c_4 \end{vmatrix} \\ + a_2c_1 \begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix} - b_2c_1 \begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix} + c_1d_2 \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} \\ - a_2d_1 \begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix} + b_2d_1 \begin{vmatrix} a_3 & c_3 \\ a_4 & c_4 \end{vmatrix} - c_2d_1 \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix}. \end{aligned}$$

The complete expansion, which may be written out directly from the above, is given in Equation (1), Article 7.

If Equation (2) of Article 27 had been used instead of Equation (1), the terms of the expansion would simply have been obtained in a different order.

2. Evaluate the determinant

$$\Delta \equiv \begin{vmatrix} 1 & 2 & 0 & 3 \\ 2 & 3 & 1 & 2 \\ 0 & 3 & 2 & 1 \\ 3 & 1 & 2 & 3 \end{vmatrix}.$$

SOLUTION. — Expanding by means of Formula (2) of Article 27, letting $s = 1$, gives

$$\Delta = 1\begin{vmatrix} 3 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + 6\begin{vmatrix} 0 & 3 \\ 2 & 3 \end{vmatrix}$$

$$- 2\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 9\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} - 9\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 12 + 3 - 3 - 16 - 36 + 12 + 18 - 54 + 27 = -37.$$

3. Evaluate the determinant

$$\Delta \equiv \begin{vmatrix} 0 & 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 & -5 \\ 8 & 0 & 8 & -5 & 0 \end{vmatrix}.$$

SOLUTION. — Applying Formulæ (1) and (2) of Article 27, remembering that the co-factor $A_k^{(s)}$ is plus or minus, according as $(k+s)$ is even or odd, gives

$$\Delta = 2\begin{vmatrix} 0 & 2 & -1 & 0 \\ 2 & -1 & 0 & 0 \\ 8 & 0 & 8 & -5 \\ 0 & 8 & -5 & 0 \end{vmatrix} + 8\begin{vmatrix} 0 & 2 & -1 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 8 & 0 & 8 & -5 \end{vmatrix}$$

$$= 10\begin{vmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ 0 & 8 & -5 \end{vmatrix} - 40\begin{vmatrix} 0 & 2 & -1 \\ 2 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix}$$

$$= -20\begin{vmatrix} 2 & -1 \\ 8 & -5 \end{vmatrix} + 40\begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} = 0.$$

4. Evaluate $\begin{vmatrix} 1 & 3 & 1 & 0 \\ 3 & 2 & 3 & 1 \\ 0 & 2 & 5 & 1 \\ 2 & 2 & 2 & 3 \end{vmatrix}$.

Ans. -85.

5. Evaluate $\begin{vmatrix} 1 & -3 & -5 & 7 \\ 7 & 1 & -3 & -5 \\ -5 & 7 & 1 & -3 \\ -3 & -5 & 7 & 1 \end{vmatrix}$.

6. Evaluate $\begin{vmatrix} 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{vmatrix}$.

7. Expand $\begin{vmatrix} a & b & c & 0 \\ a & b & 0 & d \\ a & 0 & c & d \\ 0 & b & c & d \end{vmatrix}$.

8. Show that $\begin{vmatrix} l & n' & m' & \alpha \\ n' & m & l' & \beta \\ m' & l' & n & \gamma \\ \alpha & \beta & \gamma & 0 \end{vmatrix}$

is equivalent to

$$L\alpha^2 + M\beta^2 + N\gamma^2 + 2L'\beta\gamma + 2M'\alpha\gamma + 2N'\alpha\beta,$$

in which L, M, N, L', M', N' are the co-factors of l, m, n, l', m', n' respectively in the minor

$$\begin{vmatrix} l & n' & m' \\ n' & m & l' \\ m' & l' & n \end{vmatrix}.$$

9. Develop the determinant

$$\Delta \equiv -\begin{vmatrix} 0 & \beta\gamma & \alpha\gamma & \alpha\beta \\ \beta\gamma & 1 & \cos C & \cos B \\ \alpha\gamma & \cos C & 1 & \cos A \\ \alpha\beta & \cos B & \cos A & 1 \end{vmatrix}.$$

NOTE. — The evaluation of determinants may frequently be very much simplified by the application of the theorems of Chapter III. The following examples will illustrate.

10. Evaluate the determinant

$$\Delta \equiv \begin{vmatrix} 2 & -1 & 5 & 9 \\ 3 & 3 & 3 & 11 \\ 2 & 3 & 1 & 2 \\ 5 & 7 & 3 & 7 \end{vmatrix}.$$

SOLUTION. — Adding the third column to the second (Art. 22) gives

$$\Delta = \begin{vmatrix} 2 & 4 & 5 & 9 \\ 3 & 6 & 3 & 11 \\ 2 & 4 & 1 & 2 \\ 5 & 10 & 3 & 7 \end{vmatrix},$$

whence (Art. 21)

$$\Delta = 0.$$

11. Evaluate the determinant

$$\equiv \begin{vmatrix} 2 & 7 & 6 & 5 \\ 1 & 1 & 1 & 3 \\ 1 & 5 & 3 & 4 \\ 4 & 7 & 5 & 6 \end{vmatrix}.$$

SOLUTION. — Subtracting the first column from the second and third, and three times the first from the fourth, gives

$$\Delta = \begin{vmatrix} 2 & 5 & 4 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 4 & 2 & 1 \\ 4 & 3 & 1 & -6 \end{vmatrix},$$

whence we readily obtain

$$\Delta = -\begin{vmatrix} 5 & 4 & -1 \\ 4 & 2 & 1 \\ 3 & 1 & -6 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & -1 \\ 2 & 2 & 1 \\ 2 & 1 & -6 \end{vmatrix} = -45.$$

12. Show that

$$\Delta \equiv \begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & a+c \end{vmatrix} = 4abc.$$

SOLUTION. — Subtracting the first and third rows from the second gives

$$\Delta = 2\begin{vmatrix} a+b & c & c \\ -b & 0 & -c \\ b & b & a+c \end{vmatrix}.$$

This becomes, by adding the second row to the first and third,

$$\Delta = 2\begin{vmatrix} a & c & 0 \\ -b & 0 & -c \\ 0 & b & a \end{vmatrix},$$

whence $\Delta = 4\,abc$.

13. Evaluate $\begin{vmatrix} 3 & 2 & 1 & 3 \\ 4 & 2 & 2 & 4 \\ 2 & 3 & 1 & 6 \\ 10 & 4 & 5 & 8 \end{vmatrix}$. *Ans.* 0.

14. Evaluate $\begin{vmatrix} 1 & 2 & 2 & 4 \\ 1 & 4 & 4 & 1 \\ 1 & 1 & 2 & 2 \\ 4 & 8 & 11 & 13 \end{vmatrix}$. *Ans.* 15.

15. Evaluate $\begin{vmatrix} 2 & 7 & -2 & 8 \\ 4 & 1 & 1 & -3 \\ 0 & 3 & -1 & 4 \\ 6 & 4 & 2 & -8 \end{vmatrix}$. *Ans.* 14.

16. Show that $\begin{vmatrix} \dfrac{a^2+b^2}{c} & c & c \\ a & \dfrac{b^2+c^2}{a} & a \\ b & b & \dfrac{a^2+c^2}{b} \end{vmatrix} = 4\,abc.$

17. If
$$\begin{vmatrix} a & h & g & l & 0 \\ h & b & f & m & 0 \\ g & f & c & n & 0 \\ l & m & n & 0 & t \\ 0 & 0 & 0 & t & d \end{vmatrix} = 0, \text{ show that}$$

$$d\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = t^2\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

18. Show that, if p elements of any column (or row) of a determinant are zero, while none of the elements in any of the other columns (or rows) are zero, the number of terms in the development is $(n-p)(n-1)!$

19. What is the sign of the secondary diagonal term in a determinant of the nth order?

20. What effect is produced upon a determinant of the nth order by multiplying each of its elements by (-1)? By $(-1)^{n-1}$? By $(-1)^n$? By $(-1)^{n+1}$?

21. Show that

$$\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & \gamma^2 & \beta^2 \\ b^2 & \gamma^2 & 0 & \alpha^2 \\ c^2 & \beta^2 & \alpha^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a\alpha & b\beta & c\gamma \\ a\alpha & 0 & c\gamma & b\beta \\ b\beta & c\gamma & 0 & a\alpha \\ c\gamma & b\beta & a\alpha & 0 \end{vmatrix}.$$

22. Prove that, if the sum or difference of every pair of corresponding elements of two rows (or columns) of a determinant be a constant multiple of the sum or difference of the corresponding pair of elements of two other rows (or columns), the determinant is equal to zero.

23. Show, without expansion, that the determinant

$$\begin{vmatrix} ab & c^2 & c^2 \\ a^2 & bc & a^2 \\ b^2 & b^2 & ac \end{vmatrix}$$

contains the factor $(bc + ac + ab)$.

24. Find the values of the determinants

$$\begin{vmatrix} 0 & 1 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \\ 1 & 1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \text{ and } \begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ 0 & 1 & 1 & \cdots \end{vmatrix},$$

each of the nth order.

25. If

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \text{ and } D \equiv \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_3 & d_3 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_4 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_4 & d_4 \end{vmatrix} \end{vmatrix},$$

then will $D = a_1^2\Delta$.

29. In the expansion of the determinant

$$\begin{vmatrix} a_1' & 0 & 0 & \cdots & 0 \\ a_2' & a_2'' & a_2''' & \cdots & a_2^{(n)} \\ a_3' & a_3'' & a_3''' & \cdots & a_3^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n' & a_n'' & a_n''' & \cdots & a_n^{(n)} \end{vmatrix}$$

all the terms vanish except those which contain the element a_1' from the first column (Art. 24).

It follows that we may replace the remaining elements of this column,

$$a_2', \quad a_3', \quad \ldots, \quad a_n',$$

by any quantities whatever, as

$$Q, \quad R, \quad \ldots, \quad T,$$

without changing the value of the determinant; thus,

$$\begin{vmatrix} a_1' & 0 & 0 & \cdots & 0 \\ a_2' & a_2'' & a_2''' & \cdots & a_2^{(n)} \\ a_3' & a_3'' & a_3''' & \cdots & a_3^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n' & a_n'' & a_n''' & \cdots & a_n^{(n)} \end{vmatrix} = \begin{vmatrix} a_1' & 0 & 0 & \cdots & 0 \\ Q & a_2'' & a_2''' & \cdots & a_2^{(n)} \\ R & a_3'' & a_3''' & \cdots & a_3^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ T & a_n'' & a_n''' & \cdots & a_n^{(n)} \end{vmatrix}$$

$$= a_1' \begin{vmatrix} a_2'' & a_2''' & \cdots & a_2^{(n)} \\ a_3'' & a_3''' & \cdots & a_3^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ a_n'' & a_n''' & \cdots & a_n^{(n)} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the elements a_2''', a_2^{iv}, ... $a_2^{(n)}$ also become zero, we may reduce the determinant to one of the order $(n-2)$; thus,

$$\begin{vmatrix} a_1' & 0 & 0 & \cdots & 0 \\ a_2' & a_2'' & 0 & \cdots & 0 \\ a_3' & a_3'' & a_3''' & \cdots & a_3^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n' & a_n'' & a_n''' & \cdots & a_n^{(n)} \end{vmatrix} = a_1' \begin{vmatrix} a_2'' & 0 & \cdots & 0 \\ a_3'' & a_3''' & \cdots & a_3^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ a_n'' & a_n''' & \cdots & a_n^{(n)} \end{vmatrix}$$

$$= a_1' a_2'' \begin{vmatrix} a_3''' & \cdots & a_3^{(n)} \\ \cdot & \cdot & \cdot \\ a_n''' & \cdots & a_n^{(n)} \end{vmatrix} \quad . \quad . \quad (2)$$

In this case the elements

$$\begin{matrix} a_2', & \\ a_3', & a_3'', \\ \cdots, & \cdots, \\ a_n', & a_n'', \end{matrix}$$

may be replaced by any quantities whatever, as

$$\begin{matrix} Q, & \\ R, & L, \\ \cdots, & \cdots, \\ T, & N. \end{matrix}$$

In a similar manner it may be shown that if all the elements on one side of the principal diagonal are zero, the elements on the other side of the same diagonal may be replaced by any quantities whatever, and that the determinant is equal to its principal term; thus,

$$\begin{vmatrix} a_1' & 0 & 0 & \cdots & 0 \\ a_2' & a_2'' & 0 & \cdots & 0 \\ a_3' & a_3'' & a_3''' & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n' & a_n'' & a_n''' & \cdots & a_n^{(n)} \end{vmatrix} = \begin{vmatrix} a_1' & 0 & 0 & \cdots & 0 \\ Q & a_2'' & 0 & \cdots & 0 \\ R & L & a_3''' & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ T & N & H & \cdots & a_n^{(n)} \end{vmatrix}$$

$$= a_1' \, a_2'' \, a_3''' \cdots a_n^{(n)} \quad . \ . \ . \ . \ (3)$$

30. By the preceding article we have

$$m \begin{vmatrix} a_1' & \cdots & a_1^{(n)} \\ \cdot & \cdot & \cdot \\ a_n' & \cdots & a_n^{(n)} \end{vmatrix} = \begin{vmatrix} m & 0 & \cdots & 0 \\ F & a_1' & \cdots & a_1^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ H & a_n' & \cdots & a_n^{(n)} \end{vmatrix}, \quad . \ . \ . \ (1)$$

the capitals still representing arbitrary quantities.

This furnishes another method of multiplying a determinant by a given factor.

Letting $m = 1$, we have a method of raising the order of a given determinant without changing its value.

That is

A determinant of the nth *order may be expressed as a determinant of the order* (n + 1) *by bordering it above by a row (or to the left by a column) of zeros, to the left by a column (or above by a row) of elements chosen arbitrarily, and by writing* 1 *at the intersection of the row and column thus added.*

By continuing this process, any determinant may be expressed as a determinant of any higher order; thus,

$$\begin{vmatrix} a_1' & \cdots & a_1^{(n)} \\ \cdot & \cdot & \cdot \\ a_n' & \cdots & a_n^{(n)} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ F & a_1' & \cdots & a_1^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ H & a_n' & \cdots & a_n^{(n)} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ K & 1 & 0 & \cdots & 0 \\ L & F & a_1' & \cdots & a_1^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ N & H & a_n' & \cdots & a_n^{(n)} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ P & 1 & 0 & 0 & \cdots & 0 \\ Q & K & 1 & 0 & \cdots & 0 \\ R & L & F & a_1' & \cdots & a_1^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ T & N & H & a_n' & \cdots & a_n^{(n)} \end{vmatrix} = \text{etc.} \quad . \quad . \quad (2)$$

EXAMPLES.

1. If, in any determinant Δ, a column of zeros be introduced between the sth and the $(s+1)$th columns and a row of arbitrary elements between the kth and

the $(k+1)$th rows, 1 being written at the intersection of the column and row thus added, show that the resulting determinant will be equal to $(-1)^{k+s}\Delta$.

2. Show that

$$\begin{vmatrix} a_1' & a_1'' & 0 & \cdots & 0 \\ a_2' & a_2'' & 0 & \cdots & 0 \\ 0 & 0 & a_1' & \cdots & a_1^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & a_n' & \cdots & a_n^{(n)} \end{vmatrix} = \begin{vmatrix} a_1' & a_1'' \\ a_2' & a_2'' \end{vmatrix} \times \begin{vmatrix} a_1' & \cdots & a_1^{(n)} \\ \cdot & \cdot & \cdot \\ a_n' & \cdots & a_n^{(n)} \end{vmatrix}.$$

3. Show that

$$\begin{vmatrix} 0 & 0 & 0 & \cdots & a_1^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & a_{n-2}''' & \cdots & a_{n-2}^{(n)} \\ 0 & a_{n-1}'' & a_{n-1}''' & \cdots & a_{n-1}^{(n)} \\ a_n' & a_n'' & a_n''' & \cdots & a_n^{(n)} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & \cdots & a_1^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & a_{n-2}''' & \cdots & H \\ 0 & a_{n-1}'' & L & \cdots & N \\ a_n' & Q & R & \cdots & T \end{vmatrix}$$

$$= (-1)^{\frac{n}{2}(n-1)} a_n' a_{n-1}'' a_{n-2}''' \cdots a_1^{(n)}.$$

31. *If* k *rows and* k *columns of a determinant of the* n*th order be deleted, the determinant of the order* (n − k) *formed of the* $(n-k)^2$ *remaining elements* is called a* k*th minor of the given determinant.*

---

* Deleting k rows removes kn elements; then deleting k columns removes $k(n-k)$ more, after which there remain

$$n^2 - kn - k(n-k) = (n-k)^2.$$

If the deleted rows are, in order, the hth, ith, jth, ..., and the deleted columns the pth, qth, rth, ..., the resulting minor is represented by the symbol

$$\Delta^{(p,\,q,\,r,\,\ldots}_{(h,\,i,\,j,\,\ldots)}$$

in which Δ represents the original determinant.

32. Any kth minor of a determinant, and the determinant formed of the k^2 elements at the intersections of the rows and columns deleted in forming this minor, are called, with respect to each other, complementary minors of the given determinant; thus, in the determinant

$$\Delta \equiv | \, a_1' \; a_2'' \; a_3''' \; a_4^{\text{iv}} \; a_5^{\text{v}} \, |,$$

$$\Delta^{(2,\,3,\,5}_{(1,\,3,\,5} \equiv \begin{vmatrix} a_2' & a_2^{\text{iv}} \\ a_4' & a_4^{\text{iv}} \end{vmatrix}$$

and

$$\Delta^{(1,\,4}_{(2,\,4} \equiv \begin{vmatrix} a_1'' & a_1''' & a_1^{\text{v}} \\ a_3'' & a_3''' & a_3^{\text{v}} \\ a_5'' & a_5''' & a_5^{\text{v}} \end{vmatrix}$$

are complementary minors.

It may be remarked that any element of a determinant, and the minor of the determinant with respect to that element, constitute a pair of complementary minors.

33. THEOREM. — *The product of any two complementary minors of a determinant is composed of terms which, signs being disregarded, are also found in the development of the given determinant.*

Let the given determinant, which we shall suppose to be of the nth order, be represented by Δ; also, let μ_k and μ_{n-k} represent any two complementary minors of Δ, these being of the orders k and $(n-k)$ respectively. Transpose those rows and columns of Δ which contain the elements of μ_k, so that they shall be in order the first k rows and columns of a new determinant Δ'. If the number of necessary interchanges of rows be represented by u, and of columns by v, then will (Art. 16)

$$\Delta = (-1)^{u+v}\Delta' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The determinant in its present form may be written

$$\Delta' \equiv \begin{vmatrix} a_1' & \cdots & a_1^{(k)} & \cdots & a_1^{(n)} \\ . & . & . & . & . \\ a_k' & \cdots & a_k^{(k)} & \cdots & a_k^{(n)} \\ a_{k+1}' & \cdots & \cdots & a_{k+1}^{(k+1)} & a_{k+1}^{(n)} \\ . & . & . & . & . \\ a_n' & \cdots & \cdots & a_n^{(k+1)} & a_n^{(n)} \end{vmatrix}, \quad . \quad . \quad . \quad (2)$$

in which the determinant

$$\begin{vmatrix} a_1' & \cdots & a_1^{(k)} \\ . & . & . \\ a_k' & \cdots & a_k^{(k)} \end{vmatrix},$$

occupying the upper left-hand corner of the array, is the minor μ_k referred to above, while the determinant

$$\begin{vmatrix} a_{k+1}^{(k+1)} & \cdots & a_{(k+1)}^{(n)} \\ . & . & . \\ a_n^{(k+1)} & \cdots & a_n^{(n)} \end{vmatrix},$$

occupying the lower right-hand corner, is the minor μ_{n-k}.

If the product

$$\mu_k \cdot \mu_{n-k}$$

be developed, each term of the development will be the product of a term in the development of μ_k and a term in the development of μ_{n-k}. But each term of the development of μ_k contains an element from each of the first k rows and one from each of the first k columns of Δ', and each term of the development of μ_{n-k} contains one element from each of the last $(n-k)$ rows and columns of the same determinant. Thus each term of the developed product, $\mu_k\mu_{n-k}$, contains one element from each of the n rows and columns of Δ', and is therefore a term in the development of Δ'. Hence (Eq. 1), signs being disregarded, $\mu_k\mu_{n-k}$ is composed of terms of the expansion of Δ, which was to be proven.

34. It follows from the above that, signs being duly regarded, the product

$$(-1)^{u+v}\mu_k\mu_{n-k}$$

is composed of terms of the development of Δ. Now, as may readily be seen, the only factors by which μ_k is multiplied in the development of Δ are the terms of the development of μ_{n-k} and the sign factor, $(-1)^{u+v}$. Hence, in the determinant Δ, the minor μ_k has the co-factor $(-1)^{u+v}\mu_{n-k}$. We may therefore state the following

THEOREM. — *The co-factor of any minor of a determinant is the complementary minor multiplied by the sign factor* $(-1)^{u+v}$, *in which* u *is the number of interchanges of rows, and* v *the number of interchanges of columns, necessary to bring the principal diagonals of the two minors into coincidence.*

35. If μ_k be the minor

$$\Delta\begin{matrix}(p, q, r, \ldots \\ (h, i, j, \ldots\end{matrix}$$

of the determinant Δ, the principal diagonals of μ_k and its complementary minor μ_{n-k} may be brought into coincidence by

$$(p-1)+(q-2)+(r-3)+\cdots$$

interchanges of adjacent columns, and

$$(h-1)+(i-2)+(j-3)+\cdots$$

interchanges of adjacent rows.

Hence the sign factor of the complementary minor of

$$\Delta\,{}^{(p,\,q,\,r,\ldots}_{(h,\,i,\,j,\ldots}$$

is

$$(-1)^{(p-1)+(q-2)+(r-3)+\cdots+(h-1)+(i-2)+(j-3)+\cdots},$$

which is the same as

$$(-1)^{p+q+r+\cdots+h+i+j+\cdots}.$$

36. The co-factor of a given minor of a determinant is often expressed by prefixing the syllable *co-* to the given minor; thus, in the determinant

$$\Delta \equiv |\, a_1{}' \;\; a_2{}'' \;\; \cdots \;\; a_5{}^{\mathrm{v}} \,|,$$

the co-factor of the minor

$$\begin{vmatrix} a_2{}' & a_2{}^{\mathrm{iv}} \\ a_4{}' & a_4{}^{\mathrm{iv}} \end{vmatrix}$$

may be written

$$co\text{-}\begin{vmatrix} a_2{}' & a_2{}^{\mathrm{iv}} \\ a_4{}' & a_4{}^{\mathrm{iv}} \end{vmatrix}.$$

By the two preceding articles we have

$$co\text{-}\begin{vmatrix} a_2{}' & a_2{}^{\mathrm{iv}} \\ a_4{}' & a_4{}^{\mathrm{iv}} \end{vmatrix} \equiv co\text{-}\Delta\,{}^{(2,\,3,\,5}_{(1,\,3,\,5} = (-1)^{2+3+5+1+3+5} \begin{vmatrix} a_1{}'' & a_1{}''' & a_1{}^{\mathrm{v}} \\ a_3{}'' & a_3{}''' & a_3{}^{\mathrm{v}} \\ a_5{}'' & a_5{}''' & a_5{}^{\mathrm{v}} \end{vmatrix}.$$

37. Theorem.—*If any* k *rows (or columns) of a determinant be selected and every possible minor of the order* k *be formed from them, the result obtained by multiplying each of these minors by its co-factor and taking the algebraic sum of the products, is the expansion of the given determinant.*

The given determinant being of the nth order, the number of minors of the order k which can be formed from the k rows selected is, by elementary algebra,

$$\frac{n\,!}{k\,!(n-k)\,!}.$$

Since each minor contains $k!$ terms and its co-factor $(n-k)!$ terms, the product of each minor by its co-factor contains $k!\,(n-k)!$ terms. Then, the sum of all the products which may be formed by using all the different minors contains

$$k!(n-k)! \times \frac{n\,!}{k\,!(n-k)\,!} = n\,!$$

different terms of the given determinant, which is the complete expansion.

The above method of expansion is due to Laplace. The following is an example of its application.

$$\begin{vmatrix} a_1' & a_1'' & a_1''' & a_1^{iv} \\ a_2' & a_2'' & a_2''' & a_2^{iv} \\ a_3' & a_3'' & a_3''' & a_3^{iv} \\ a_4' & a_4'' & a_4''' & a_4^{iv} \end{vmatrix} = \begin{vmatrix} a_1' & a_1'' \\ a_2' & a_2'' \end{vmatrix} \begin{vmatrix} a_3''' & a_3^{iv} \\ a_4''' & a_4^{iv} \end{vmatrix} - \begin{vmatrix} a_1' & a_1''' \\ a_2' & a_2''' \end{vmatrix} \begin{vmatrix} a_3'' & a_3^{iv} \\ a_4'' & a_4^{iv} \end{vmatrix}$$

$$+ \begin{vmatrix} a_1' & a_1^{iv} \\ a_2' & a_2^{iv} \end{vmatrix} \begin{vmatrix} a_3'' & a_3''' \\ a_4'' & a_4''' \end{vmatrix}$$

$$+ \begin{vmatrix} a_1'' & a_1''' \\ a_2'' & a_2''' \end{vmatrix} \begin{vmatrix} a_3' & a_3^{iv} \\ a_4' & a_4^{iv} \end{vmatrix}$$

$$- \begin{vmatrix} a_1'' & a_1^{iv} \\ a_2'' & a_2^{iv} \end{vmatrix} \begin{vmatrix} a_3' & a_3''' \\ a_4' & a_4''' \end{vmatrix} + \begin{vmatrix} a_1''' & a_1^{iv} \\ a_2''' & a_2^{iv} \end{vmatrix} \begin{vmatrix} a_3' & a_3'' \\ a_4' & a_4'' \end{vmatrix}.$$

Formulæ (1) and (2) of Article 28 are special cases of the application of Laplace's theorem.

38. It is often necessary to expand a determinant in terms of the elements of a given row and column. This may be done by means of a formula due to Cauchy, which we now proceed to develop.

Let the determinant be represented by Δ, the given row and column being the hth and the pth, respectively; then will $A_h^{(p)}$ be the *co-factor* and $\Delta_{(h}^{(p}$ the *complementary minor* of the element $a_h^{(p)}$ at the intersection of the given row and column. Also, let $B_k^{(s)}$ be the *co-factor* of any element $a_k^{(s)}$ of the minor $\Delta_{(h}^{(p}$.

All the terms of the development of Δ which contain the element $a_h^{(p)}$ are found in the product

$$a_h^{(p)} A_h^{(p)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

All the terms which contain the elements $a_h^{(s)}$ from the hth row and $a_k^{(p)}$ from the pth column, are found in the product (Art. 35)

$$(-1)^{h+k+p+s} \cdot \begin{vmatrix} a_h^{(p)} & a_h^{(s)} \\ a_k^{(p)} & a_k^{(s)} \end{vmatrix} \Delta\left\{\begin{smallmatrix} p, & s \\ h, & k \end{smallmatrix}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

and consequently in

$$-(-1)^{h+k+p+s} \cdot a_k^{(p)} a_h^{(s)} \Delta\left\{\begin{smallmatrix} p, & s \\ h, & k \end{smallmatrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

which is equal to (Art. 26)

$$-(-1)^{h+p} a_k^{(p)} a_h^{(s)} B_k^{(s)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

The remaining term,

$$(-1)^{h+k+p+s} a_h^{(p)} a_k^{(s)} \Delta\left\{\begin{smallmatrix} p, & s \\ h, & k \end{smallmatrix}\right\}$$

of the product (2) is included in the product (1) and must not again be introduced in the expansion.

Forming all the products like (4) which can be formed by taking each time one element from the hth row and one from the pth column, the complete expansion may be written

$$\Delta = a_h^{(p)} A_h^{(p)} - \Sigma(-1)^{h+p} a_k^{(p)} a_h^{(s)} B_k^{(s)} \quad . \quad . \qquad (5)$$

This is Cauchy's formula.

If it be required to expand the determinant in terms of the elements of the first row and the first column, the above formula becomes

$$\Delta = a_1' A_1' - \Sigma a_k' a_1^{(s)} B_k^{(s)}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

or, more explicitly,

$$\begin{aligned}\Delta = a_1' A_1' - a_2'a_1''B_2'' - a_2'a_1'''B_2''' \cdots - a_2'a_1^{(n)}B_2^{(n)}\\ - a_3'a_1''B_3'' - a_3'a_1'''B_3''' \cdots - a_3'a_1^{(n)}B_3^{(n)}\\ \cdots\cdots\cdots\cdots\cdots\\ - a_n'a_1''B_n'' - a_n'a_1'''B_n''' \cdots - a_n'a_1^{(n)}B_n^{(n)},\end{aligned}$$

in which $B_k^{(s)}$ has the sign

$$(-1)^{k+s} \qquad \qquad (7)$$

Cauchy's method is useful in expanding determinants which have been bordered, a class of determinants quite common in analysis. We take, as an example, the bordered determinant

$$\Delta \equiv \begin{vmatrix} 1 & l & m & n \\ x & a_1' & a_1'' & a_1''' \\ y & a_2' & a_2'' & a_2''' \\ z & a_3' & a_3'' & a_3''' \end{vmatrix}.$$

The expansion by Formula (6), or (7), is

$$\begin{aligned}\Delta = \begin{vmatrix} a_1' & a_1'' & a_1''' \\ a_2' & a_2'' & a_2''' \\ a_3' & a_3'' & a_3''' \end{vmatrix} - lx \begin{vmatrix} a_2'' & a_2''' \\ a_3'' & a_3''' \end{vmatrix} + mx \begin{vmatrix} a_2' & a_2''' \\ a_3' & a_3''' \end{vmatrix} - nx \begin{vmatrix} a_2' & a_2'' \\ a_3' & a_3'' \end{vmatrix}\\ + ly \begin{vmatrix} a_1'' & a_1''' \\ a_3'' & a_3''' \end{vmatrix} - my \begin{vmatrix} a_1' & a_1''' \\ a_3' & a_3''' \end{vmatrix} + ny \begin{vmatrix} a_1' & a_1'' \\ a_3' & a_3'' \end{vmatrix}\\ - lz \begin{vmatrix} a_1'' & a_1''' \\ a_2'' & a_2''' \end{vmatrix} + mz \begin{vmatrix} a_1' & a_1''' \\ a_2' & a_2''' \end{vmatrix} - nz \begin{vmatrix} a_1' & a_1'' \\ a_2' & a_2'' \end{vmatrix}.\end{aligned}$$

EXAMPLES.

1. Expand the determinant in Example 8, after Article 28, by Cauchy's method.

2. Expand the determinant in Example 9, after Article 28, by Cauchy's method.

3. If a determinant of the nth order be divided into two sets of n and $(n-m)$ rows, and if there be n columns of zeros in the set of $(n-m)$ rows, the determinant is equal to the product of the minor of the remaining $(n-m)$ columns and its complementary minor.

4. Prove that, in a determinant of the nth order, if a set of m rows contains more than $(n-m)$ columns of zeros the determinant vanishes.

5. Expand the determinant

$$\begin{vmatrix} 0 & -b & -c & -d \\ b & 0 & -e & -f \\ c & e & 0 & -g \\ d & f & g & 0 \end{vmatrix}.$$

6. Expand $\begin{vmatrix} x & 1 & 1 & 1 \\ -1 & x^3 & 1 & 1 \\ -1 & -1 & x^3 & 1 \\ -1 & -1 & -1 & x \end{vmatrix}.$

7. Expand $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix}$.

8. Expand $\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix}$.

9. Expand $\begin{vmatrix} 0 & x & y & z \\ x & 1 & 2 & 3 \\ y & 2 & 1 & 4 \\ z & 3 & 4 & 1 \end{vmatrix}$.

10. Develop the determinant

$$\begin{vmatrix} 0 & a_1'' & a_1''' & \cdots & \cdots & a_1^{(n)} \\ -a_1'' & 0 & a_2''' & \cdots & \cdots & a_2^{(n)} \\ -a_1''' & -a_2''' & 0 & \cdots & \cdots & a_3^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & 0 & a_{n-1}^{(n)} \\ -a_1^{(n)} & -a_2^{(n)} & -a_3^{(n)} & \cdots & -a_{n-1}^{(n)} & 0 \end{vmatrix}$$

according to the elements of the first column and the first row.

39.* Let it be required to find the differential of the determinant

$$\Delta \equiv \begin{vmatrix} x_1' & x_1' & \cdots & x_1^{(n)} \\ x_2' & x_2'' & \cdots & x_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ x_n' & x_n'' & \cdots & x_n^{(n)} \end{vmatrix}.$$

By Formula (1) of Article 27,

$$\Delta = X_k'x_k' + X_k''x_k'' + \cdots + X_k^{(n)}x_k^{(n)}.$$

The differential of this with respect to the element $x_k^{(s)}$ is

$$d_{(k)}^{(s)}\Delta = X_k^{(s)}dx_k^{(s)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that is, *the differential of a determinant with respect to any element is the co-factor of that element multiplied by the differential of the element.*

The total differential *taken with respect to the elements of the* k*th row* is the sum of the partial differentials with respect to the elements of the kth row: that is,

$$d_{(k)}\Delta = X_k'dx_k' + X_k''dx_k'' + \cdots + X_k^{(n)}dx_k^{(n)},$$

or,
$$d_{(k)}\Delta = \begin{vmatrix} x_1' & x_1'' & \cdots & x_1^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ x_{k-1}' & x_{k-1}'' & \cdots & x_{k-1}^{(n)} \\ dx_k' & dx_k'' & \cdots & dx_k^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ x_n' & x_n'' & \cdots & x_n^{(n)} \end{vmatrix} \quad . \quad . \quad . \quad (2)$$

* The reader who is not familiar with the differential calculus may omit this article.

The total differential *taken with respect to all the elements of the determinant* is

$$d\Delta = d_{(1)}\Delta + d_{(2)}\Delta + \cdots + d_{(n)}\Delta,$$

or, by the preceding equation,

$$d\Delta = \begin{vmatrix} dx_1' & dx_1'' & \cdots & dx_1^{(n)} \\ x_2' & x_2'' & \cdots & x_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ x_n' & x_n'' & \cdots & x_n^{(n)} \end{vmatrix} + \begin{vmatrix} x_1' & x_1'' & \cdots & x_1^{(n)} \\ dx_2' & dx_2'' & \cdots & dx_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ x_n' & x_n'' & \cdots & x_n^{(n)} \end{vmatrix} +$$

$$\cdots + \begin{vmatrix} x_1' & x_1'' & \cdots & x_1^{(n)} \\ x_2' & x_2'' & \cdots & x_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ dx_n' & dx_n'' & \cdots & dx_n^{(n)} \end{vmatrix}. \quad . \quad (3)$$

In the same manner we may derive the formula

$$d\Delta = \begin{vmatrix} dx_1' & x_1'' & \cdots & x_1^{(n)} \\ dx_2' & x_2'' & \cdots & x_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ dx_n' & x_n'' & \cdots & x_n^{(n)} \end{vmatrix} + \begin{vmatrix} x_1' & dx_1'' & \cdots & x_1^{(n)} \\ x_2' & dx_2'' & \cdots & x_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ x_n' & dx_n'' & \cdots & x_n^{(n)} \end{vmatrix} +$$

$$\cdots + \begin{vmatrix} x_1' & x_1'' & \cdots & dx_1^{(n)} \\ x_2' & x_2'' & \cdots & dx_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ x_n' & x_n'' & \cdots & dx_n^{(n)} \end{vmatrix}. \quad . \quad (4)$$

EXAMPLES.

1. Given $\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$; find $\dfrac{dy}{dx}$.

Ans. $\dfrac{b_1 - b_2}{a_1 - a_2}$.

2. Differentiate $\begin{vmatrix} \sin x & \sin y \\ \cos x & \cos y \end{vmatrix}$.

3. Differentiate $\begin{vmatrix} 1 & x & 1 & y \\ x & 1 & y & 1 \\ 1 & y & 1 & x \\ y & 1 & x & 1 \end{vmatrix}$.

4. Differentiate $\begin{vmatrix} 0 & yz & xz & xy \\ yz & 1 & 1 & 1 \\ xz & 1 & 1 & 1 \\ xy & 1 & 1 & 1 \end{vmatrix}$.

5. Differentiate $\begin{vmatrix} 1 & \cos z & \cos y \\ \cos z & 1 & \cos x \\ \cos y & \cos x & 1 \end{vmatrix}$.

6. Given $\Delta = \begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & 0 \end{vmatrix}$;

show that
$$\frac{d\Delta}{dx} = -2(Ax + Hy + Gz),$$

$$\frac{d\Delta}{dy} = -2(Hx + By + Fz),$$

and
$$\frac{d\Delta}{dz} = -2(Gx + Fy + Cz),$$

A, B, C, F, G, and H being co-factors of the elements of

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

CHAPTER V.

APPLICATIONS OF DETERMINANTS TO ELEMENTARY ALGEBRA.

Enough of the theory of determinants has now been developed to enable us to study, by its means, some of the fundamental properties of systems of equations. It is with this application of the theory that the present chapter is principally concerned.

40. Let it be required to solve the simultaneous linear equations

$$a_1'x' + a_1''x'' + a_1'''x''' = k_1,$$
$$a_2'x' + a_2''x'' + a_2'''x''' = k_2,$$
$$a_3'x' + a_3''x'' + a_3'''x''' = k_3.$$

The above equations holding good, we may write, arbitrarily, the equation

$$\begin{vmatrix} (a_1'x' + a_1''x'' + a_1'''x''') & a_1'' & a_1''' \\ (a_2'x' + a_2''x'' + a_2'''x''') & a_2'' & a_2''' \\ (a_3'x' + a_3''x'' + a_3'''x''') & a_3'' & a_3''' \end{vmatrix} = \begin{vmatrix} k_1 & a_1'' & a_1''' \\ k_2 & a_2'' & a_2''' \\ k_3 & a_3'' & a_3''' \end{vmatrix}.$$

By Articles 19 and 20 the first member of this equation may be written

$$x' \begin{vmatrix} a_1' & a_1'' & a_1''' \\ a_2' & a_2'' & a_2''' \\ a_3' & a_3'' & a_3''' \end{vmatrix} + x'' \begin{vmatrix} a_1'' & a_1'' & a_1''' \\ a_2'' & a_2'' & a_2''' \\ a_3'' & a_3'' & a_3''' \end{vmatrix} + x''' \begin{vmatrix} a_1''' & a_1'' & a_1''' \\ a_2''' & a_2'' & a_2''' \\ a_3''' & a_3'' & a_3''' \end{vmatrix},$$

each term of which, except the first, vanishes by Article 18. The preceding equation thus becomes

$$x' \begin{vmatrix} a_1' & a_1'' & a_1''' \\ a_2' & a_2'' & a_2''' \\ a_3' & a_3'' & a_3''' \end{vmatrix} = \begin{vmatrix} k_1 & a_1'' & a_1''' \\ k_2 & a_2'' & a_2''' \\ k_3 & a_3'' & k_3''' \end{vmatrix}, \text{ or}$$

$$x' = \frac{\begin{vmatrix} k_1 & a_1'' & a_1''' \\ k_2 & a_2'' & a_2''' \\ k_3 & a_3'' & a_3''' \end{vmatrix}}{\begin{vmatrix} a_1' & a_1'' & a_1''' \\ a_2' & a_2'' & a_2''' \\ a_3' & a_3'' & a_3''' \end{vmatrix}}.$$

The values of x'' and x''' may be found in the same manner. This method is readily seen to be a general one for the solution of systems of linear equations.

Let us now proceed, in exactly the same manner as above, to solve the following system of n simultaneous linear equations, involving the n unknown quantities, x', x'', x''', $\cdots$ $x^{(n)}$:

$$\left.\begin{array}{l}
a_1'x' + a_1''x'' + \cdots + a_1^{(i-1)}x^{(i-1)} + a_1^{(i)}x^{(i)} + a_1^{(i+1)}x^{(i+1)} \\
\qquad\qquad + \cdots + a_1^{(n)}x^{(n)} = k_1, \\
a_2'x' + a_2''x'' + \cdots + a_2^{(i-1)}x^{(i-1)} + a_2^{(i)}x^{(i)} + a_2^{(i+1)}x^{(i+1)} \\
\qquad\qquad + \cdots + a_2^{(n)}x^{(n)} = k_2, \\
a_3'x' + a_3''x'' + \cdots + a_3^{(i-1)}x^{(i-1)} + a_3^{(i)}x^{(i)} + a_3^{(i+1)}x^{(i+1)} \\
\qquad\qquad + \cdots + a_3^{(n)}x^{(n)} = k_3, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_n'x' + a_n''x'' + \cdots + a_n^{(i-1)}x^{(i-1)} + a_n^{(i)}x^{(i)} + a_n^{(i+1)}x^{(i+1)} \\
\qquad\qquad + \cdots + a_n^{(n)}x^{(n)} = k_n.
\end{array}\right\} \quad (1)$$

Let $x^{(i)}$ be the unknown whose value is sought. The above equations holding good, we may write, arbitrarily, the equation

$$\begin{vmatrix}
a_1'a_1''\cdots a_1^{(i-1)}(a_1'x' + \cdots + a_1^{(i)}x^{(i)} + \cdots + a_1^{(n)}x^{(n)})\,a_1^{(i+1)}\cdots a_1^{(n)} \\
a_2'a_2''\cdots a_2^{(i-1)}(a_2'x' + \cdots + a_2^{(i)}x^{(i)} + \cdots + a_2^{(n)}x^{(n)})\,a_2^{(i+1)}\cdots a_2^{(n)} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_n'a_n''\cdots a_n^{(i-1)}(a_n'x' + \cdots + a_n^{(i)}x^{(i)} + \cdots + a_n^{(n)}x^{(n)})\,a_n^{(i+1)}\cdots a_n^{(n)}
\end{vmatrix}$$

$$= \begin{vmatrix}
a_1'a_1''\cdots a_1^{(i-1)}k_1a_1^{(i+1)}\cdots a_1^{(n)} \\
a_2'a_2''\cdots a_2^{(i-1)}k_2a_2^{(i+1)}\cdots a_2^{(n)} \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_n'a_n''\cdots a_n^{(i-1)}k_na_n^{(i+1)}\cdots a_n^{(n)}
\end{vmatrix} \quad . \quad . \quad . \quad . \quad (2)$$

By Articles 19 and 20 the first member of this equation may be written

$$x'\begin{vmatrix} a_1'a_1'' & \cdots & a_1^{(i-1)}a_1' & a_1^{(i+1)} & \cdots & a_1^{(n)} \\ a_2'a_2'' & \cdots & a_2^{(i-1)}a_2' & a_2^{(i+1)} & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n'a_n'' & \cdots & a_n^{(i-1)}a_n' & a_n^{(i+1)} & \cdots & a_n^{(n)} \end{vmatrix}$$

$$+\cdots+x^{(i)}\begin{vmatrix} a_1'a_1'' & \cdots & a_1^{(i-1)}a_1^{(i)} & a_1^{(i+1)} & \cdots & a_1^{(n)} \\ a_2'a_2'' & \cdots & a_2^{(i-1)}a_2^{(i)} & a_2^{(i+1)} & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n'a_n'' & \cdots & a_n^{(i-1)}a_n^{(i)} & a_n^{(i+1)} & \cdots & a_n^{(n)} \end{vmatrix}$$

$$+\cdots+x^{(n)}\begin{vmatrix} a_1'a_1'' & \cdots & a_1^{(i-1)}a_1^{(n)} & a_1^{(i+1)} & \cdots & a_1^{(n)} \\ a_2'a_2'' & \cdots & a_2^{(i-1)}a_2^{(n)} & a_2^{(i+1)} & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n'a_n'' & \cdots & a_n^{(i-1)}a_n^{(n)} & a_n^{(i+1)} & \cdots & a_n^{(n)} \end{vmatrix}.$$

Each term of the above expression, with the exception of the one containing $x^{(i)}$, vanishes by Article 18. Hence Equation (2) becomes

$$x^{(i)}\begin{vmatrix} a_1'a_1'' & \cdots & a_1^{(i-1)}a_1^{(i)} & a_1^{(i+1)} & \cdots & a_1^{(n)} \\ a_2'a_2'' & \cdots & a_2^{(i-1)}a_2^{(i)} & a_2^{(i+1)} & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n'a_n'' & \cdots & a_n^{(i-1)}a_n^{(i)} & a_n^{(i+1)} & \cdots & a_n^{(n)} \end{vmatrix}$$

$$=\begin{vmatrix} a_1'a_1'' & \cdots & a_1^{(i-1)}k_1 & a_1^{(i+1)} & \cdots & a_1^{(n)} \\ a_2'a_2'' & \cdots & a_2^{(i-1)}k_2 & a_2^{(i+1)} & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n'a_n'' & \cdots & a_n^{(i-1)}k_n & a_n^{(i+1)} & \cdots & a_n^{(n)} \end{vmatrix},$$

$$\text{or} \qquad x^{(i)} = \frac{\begin{vmatrix} a_1'a_1'' \cdots a_1^{(i-1)}k_1 & a_1^{(i+1)} \cdots a_1^{(n)} \\ a_2'a_2'' \cdots a_2^{(i-1)}k_2 & a_2^{(i+1)} \cdots a_2^{(n)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'a_n'' \cdots a_n^{(i-1)}k_n & a_n^{(i+1)} \cdots a_n^{(n)} \end{vmatrix}}{\begin{vmatrix} a_1'a_1'' \cdots a_1^{(i-1)}a_1^{(i)} & a_1^{(i+1)} \cdots a_1^{(n)} \\ a_2'a_2'' \cdots a_2^{(i-1)}a_2^{(i)} & a_2^{(i+1)} \cdots a_2^{(n)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'a_n'' \cdots a_n^{(i-1)}a_n^{(i)} & a_n^{(i+1)} \cdots a_n^{(n)} \end{vmatrix}} \quad . \; . \; . \quad (3)$$

An inspection of Equation (3) enables us to state the following

THEOREM. — (a) *The common denominator of the fractions expressing the values of the unknowns in a system of* n *linear equations involving* n *unknown quantities is the determinant of the coefficients.* (b) *The numerator of the fraction expressing the value of any one of the unknowns, is a determinant which may be formed from the determinant of the coefficients by substituting for the column containing the coefficients of this unknown a column whose elements are the absolute terms of the equations, these being taken in the same order as the coefficients which they displace.*

In Articles 3 and 6 we have already met with special cases of the above theorem.

EXAMPLES.

1. Solve the simultaneous equations,

$$-x+y+z+w=8,$$
$$x-y+z+w=6,$$
$$x+y-z+w=4,$$
and
$$x+y+z-w=2.$$

2. Solve the simultaneous equations,

$$x+y=4,$$
$$y+z=8,$$
$$z+w=12,$$
and
$$w+x=8.$$

3. Solve the system of equations,

$$x+2y-z=0,$$
$$y+z-2w=0,$$
$$-x+z+2w=25,$$
and
$$2x-5y+3w=-\tfrac{5}{2}.$$

4. Find the values of the unknowns in the system,

$$x+y+z=0,$$
$$3x+2y+z=0,$$
and
$$2y+3z=0.$$

5. Solve the system,

$$x+y-z=0,$$

$$x+4y-3z=0,$$

and $$5x-y-z=0. \quad \text{(See Art. 42.)}$$

6. Representing the expansion of the rational fraction

$$\frac{a_0+a_1x+a_2x^2+a_3x^3+\cdots}{b_0+b_1x+b_2x^2+b_3x^3+\cdots},$$

in ascending powers of x, by $q_0+q_1x+q_2x^2+q_3x^3+\cdots$, show that

$$q_n=\frac{1}{b_0^{n+1}}\begin{vmatrix} b_0 & 0 & 0 & \cdots 0 & a_0 \\ b_1 & b_0 & 0 & \cdots 0 & a_1 \\ b_2 & b_1 & b_0 & \cdots 0 & a_2 \\ b_3 & b_2 & b_1 & \cdots 0 & a_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_n & b_{n-1} & b_{n-2} & \cdots b_1 & a_n \end{vmatrix}.$$

Suggestion. — Clear of fractions and equate coefficients.

41. If the number of equations in a given system be greater than the number of unknowns, it will not, in general, be possible to assign to the unknowns values which will simultaneously satisfy all the given equations. Whenever values may be assigned to the unknowns which will simultaneously satisfy all the equations, the system is said to be **consistent.**

The consistency of any system containing a redundance of equations must obviously depend upon some relation among the coefficients. We now proceed to investigate this relation in the case of $(n+1)$ linear equations involving n unknowns.

In this case the equations may be written,

$$\left.\begin{array}{l} a_1'x' \;\; + a_1''x'' \;\; + \cdots + a_1^{(n)}x^{(n)} \;\;\; = k_1 \\ a_2'x' \;\; + a_2''x'' \;\; + \cdots + a_2^{(n)}x^{(n)} \;\;\; = k_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'x' \;\; + a_n''x'' \;\; + \cdots + a_n^{(n)}x^{(n)} \;\;\; = k_n \\ a_{n+1}'x' + a_{n+1}''x'' + \cdots + a_{n+1}^{(n)}x^{(n)} = k_{n+1} \end{array}\right\} \ldots (1)$$

Since the above system is to be regarded as consistent, the values of the unknowns obtained by solving any n of the equations must satisfy the remaining equation.

Solving the last n equations by the method explained in Article 40, we obtain

$$x' = \frac{\begin{vmatrix} k_2 & a_2'' & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ k_{n+1} & a_{n+1}'' & \cdots & a_{n+1}^{(n)} \end{vmatrix}}{\begin{vmatrix} a_2' & \cdots & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1}' & \cdots & \cdots & a_{n+1}^{(n)} \end{vmatrix}} = (-1)^{n-1} \frac{\begin{vmatrix} a_2'' & \cdots & a_2^{(n)} & k_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1}'' & \cdots & a_{n+1}^{(n)} & k_{n+1} \end{vmatrix}}{\begin{vmatrix} a_2' & \cdots & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1}' & \cdots & \cdots & a_{n+1}^{(n)} \end{vmatrix}},$$

$$x'' = (-1)^{n-2} \frac{\begin{vmatrix} a_2' & a_2''' & \cdots & a_2^{(n)} & k_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n+1}' & a_{n+1}''' & \cdots & a_{n+1}^{(n)} & k_{n+1} \end{vmatrix}}{\begin{vmatrix} a_2' & \cdots & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1}' & \cdots & \cdots & a_{n+1}^{(n)} \end{vmatrix}},$$

. . . .

$$x^{(n)} = \frac{\begin{vmatrix} a_2' & \cdots & a_2^{(n-1)} & k_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1}' & \cdots & a_{n+1}^{(n-1)} & k_{n+1} \end{vmatrix}}{\begin{vmatrix} a_2' & \cdots & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1}' & \cdots & \cdots & a_{n+1}^{(n)} \end{vmatrix}}.$$

Substituting these values in the first of Equations (1) and clearing of fractions, we obtain by the aid of Formula (1) of Article 27,

$$\begin{vmatrix} a_1' & \cdots & a_1^{(n)} & k_1 \\ a_2' & \cdots & a_2^{(n)} & k_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_n' & \cdots & a_n^{(n)} & k_n \\ a_{n+1}' & \cdots & a_{n+1}^{(n)} & k_{n+1} \end{vmatrix} = 0, \quad . \quad . \quad . \quad (2)$$

which is the required relation among the coefficients; that is, the condition of consistency.

Hence, *the condition of consistency for a system of linear equations involving a number of unknowns one less than the number of equations is that the determinant of the coefficients and absolute terms shall be equal to zero.*

When the equations are consistent, this determinant is called the **eliminant** or **resultant** of the system, because it is the result obtained by eliminating the unknowns from the given equations.

42. The case of a system of n homogeneous linear equations involving n unknowns is closely related to that just considered, but demands special consideration.

In this case the column of absolute terms (k's) becomes a column of zeros, and the numerators of the fractions giving the values of the unknowns vanish. (See Arts. 40 and 20.)

Hence a system of homogeneous linear equations will always be satisfied by giving to each unknown the value zero.*

Example 4 after Article 40 furnishes a case in point.

There are, however, systems of this kind in which the equations may be simultaneously satisfied by assigning to the unknowns values other than zero. These cases we shall now consider.

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* This is, of course, obvious on other grounds, and is true for homogeneous equations of any degree.

Let

$$\left.\begin{array}{l} a_1'x' + a_1''x'' + \cdots + a_1^{(n)}x^{(n)} = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'x' + a_n''x'' + \cdots + a_n^{(n)}x^{(n)} = 0 \end{array}\right\} \quad . \quad . \quad (1)$$

be any system of n homogeneous linear equations involving the n unknowns $x', x'', \cdots x^{(n)}$, in which the coefficients are so related that

$$\begin{vmatrix} a_1' & a_1'' \cdots a_1^{(n)} \\ \cdot & \cdot \quad \cdot \quad \cdot \\ a_n' & a_n'' \cdots a_n^{(n)} \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad (2)$$

Applying the method of Article 40 to the above system, we can obtain the value of each unknown only in the form $\frac{0}{0}$, which may have any value whatever.

Though it is thus impossible to determine the absolute values of the unknowns in a system such as the above, it is possible to find the ratios of any $(n-1)$ of the unknowns to the remaining one. For, dividing each of Equations (1) by $x^{(i)}$ and representing the ratios

$$\frac{x'}{x^{(i)}}, \ \frac{x''}{x^{(i)}}, \ \cdots \ \frac{x^{(n)}}{x^{(i)}} \text{ by } v', \ v'', \cdots v^{(n)},$$

respectively, remembering that $v^{(i)} = 1$, we obtain the system of equations

$$\left.\begin{array}{l} a_1'v' + \cdots + a_1^{(i-1)}v^{(i-1)} + a_1^{(i+1)}v^{(i+1)} + \cdots + a_1^{(n)}v^{(n)} = -a_1^{(i)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'v' + \cdots + a_n^{(i-1)}v^{(i-1)} + a_n^{(i+1)}v^{(i+1)} + \cdots + a_n^{(n)}v^{(n)} = -a_n^{(i)} \end{array}\right\} (3)$$

This is a system of n non-homogeneous linear equations involving $(n-1)$ unknowns, $v', \cdots v^{(i-1)}$, $v^{(i+1)}, \cdots v^{(n)}$, and the condition expressed by Equation (2) holding good, the system is consistent, and the values of the ratios $v', \cdots v^{(i-1)}, v^{(i+1)}, \cdots v^{(n)}$ may generally* be obtained by solving any $(n-1)$ of Equations (3). Hence, Equations (3), and consequently Equations (1), will be satisfied by any values of $x', x'', \cdots x^{(n)}$ among which we have the ratios $v', v'', \cdots v^{(n)}$, as determined by any $(n-1)$ of Equations (3); that is, if Equations (1) are satisfied by the values $x_0', x_0'', \cdots x_0^{(n)}$, they will also be satisfied by the values $\lambda x_0', \lambda x_0'', \cdots \lambda x_0^{(n)}$, λ being any factor whatever.

Since only $(n-1)$ of the n equations (3) are required to determine $v', v'', \cdots v^{(n)}$, it follows that in any system of $(n-1)$ homogeneous linear equations involving n unknowns, the values of the ratios of these unknowns may be determined.

If there are n equations in the system, the vanishing of the determinant of the coefficient, shows that

* The only limitation is that there shall be at least one element in the determinant (2) whose co-factor does not vanish.

the same values of the ratios $v', v'', \dots v^{(n)}$ may be deduced from any $(n-1)$ of the equations; that is, that the equations are *consistent.*

Hence, *the condition of consistency for a system of* n *homogeneous linear equations involving* n *unknowns is that the determinant of the coefficients shall be equal to zero.*

When such a system is consistent, the determinant of the coefficients is called the *eliminant* or *resultant* of the system; thus, the determinant (2) is the eliminant of the system (1).

By way of illustration, we resume Example 5, after Article 40. We have

$$\begin{vmatrix} 1 & 1 & -1 \\ 1 & 4 & -3 \\ 5 & -1 & -1 \end{vmatrix} = 0;$$

that is, the determinant of the coefficients vanishes. This gives (Art. 40)

$$x = \frac{0}{0}, \quad y = \frac{0}{0}, \quad z = \frac{0}{0},$$

results which the student has already obtained; hence the equations fail to determine the values of the unknowns.

The vanishing of the above determinant, however, shows the system to be consistent.

To find the values of the ratios $\frac{x}{z}$ and $\frac{y}{z}$, let the equations be divided by z. They thus become

$$\left.\begin{aligned} \frac{x}{z} + \frac{y}{z} &= 1 \\ \frac{x}{z} + 4\frac{y}{z} &= 3 \\ 5\frac{x}{z} - \frac{y}{z} &= 1 \end{aligned}\right\},$$

any two of which give

$$\frac{x}{z} = \frac{1}{3}, \quad \frac{y}{z} = \frac{2}{3}, \quad \text{or} \quad x : y : z :: 1 : 2 : 3,$$

and any quantities having these ratios will satisfy the given equations.

43. Having given the relation

$$\Delta \equiv \begin{vmatrix} a_1' & a_1'' \cdots & a_1^{(n)} \\ \cdot & \cdot \quad \cdot & \cdot \\ a_n' & a_n'' \cdots & a_n^{(n)} \end{vmatrix} = 0,$$

then, by Articles 27 and 28, we may write

$$\left.\begin{array}{l} a_1'A_k' + a_1''A_k'' + \cdots + a_1^{(n)}A_k^{(n)} = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_k'A_k' + a_k''A_k'' + \cdots + a_k^{(n)}A_k^{(n)} = 0 = \Delta \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'A_k' + a_n''A_k'' + \cdots + a_n^{(n)}A_k^{(n)} = 0 \end{array}\right\},$$

in which the values of the ratios

$$\frac{A_k'}{A_k^{(i)}}, \quad \cdots \quad \frac{A_k^{(i-1)}}{A_k^{(i)}}, \quad \frac{A_k^{(i+1)}}{A_k^{(i)}}, \quad \cdots \quad \frac{A_k^{(n)}}{A_k^{(i)}},$$

no matter which subscript k may be, are the same as the values of the ratios

$$\frac{x'}{x^{(i)}}, \quad \cdots \quad \frac{x^{(i-1)}}{x^{(i)}}, \quad \frac{x^{(i+1)}}{x^{(i)}}, \quad \cdots \quad \frac{x^{(n)}}{x^{(i)}},$$

given by the equations

$$\left.\begin{array}{l} a_1'x' + a_1''x'' + \cdots + a_1^{(n)}x^{(n)} = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'x' + a_n''x'' + \cdots + a_n^{(n)}x^{(n)} = 0 \end{array}\right\} \quad . \quad . \quad . \quad (1)$$

Therefore $x', x'', \cdots x^{(n)}$ are proportional to A_k', A_k'', $\cdots A_k^{(n)}$, and giving to k the values $1, 2, \cdots n$ in succession, we have

$$\left.\begin{array}{rl} x' : x'' : \cdots : x^{(n)} & :: A_1' : A_1'' : \cdots : A_1^{(n)} \\ & :: A_2' : A_2'' : \cdots : A_2^{(n)} \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & :: A_n' : A_n'' : \cdots : A_n^{(n)} \end{array}\right\} \quad . \quad . \quad . (2)$$

Hence, *in any determinant which equals zero, the co-factors of the elements in any row (or column) are proportional to the co-factors of the corresponding elements in any other row (or column).*

44. Of the proportions (2) in the preceding article let us consider the last, viz.:

$$x' : x'' : \cdots : x^{(n)} :: A_n' : A_n'' : \cdots : A_n^{(n)}. \quad . \quad . \quad (1)$$

The coefficients of the last of Equations (1) of the same article do not appear in the co-factors A_n', A_n'', $\cdots$ $A_n^{(n)}$, and it follows that the proportions (1) give the ratios of the unknowns x', x'', $\cdots$ $x^{(n)}$ which satisfy the $(n-1)$ equations

$$\left.\begin{array}{l} a_1'x' \quad + a_1''x'' \quad + \cdots + a_1^{(n)}x^{(n)} \quad = 0 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ a_{n-1}'x' + a_{n-1}''x'' + \cdots + a_{n-1}^{(n)}x^{(n)} = 0 \end{array}\right\}, \quad . \quad . \quad (2)$$

expressed in terms of determinants formed from the coefficients of these equations by suppressing the column of coefficients of each unknown in turn; thus, if we place

$$D^{(i)} = (-1)^{(i-1)} \begin{vmatrix} a_1' & \cdots & a_1^{(i-1)} & a_1^{(i+1)} & \cdots & a_1^{(n)} \\ . & . & . & . & . & . \\ a_{n-1}' & \cdots & a_{n-1}^{(i-1)} & a_{n-1}^{(i+1)} & \cdots & a_{n-1}^{(n)} \end{vmatrix}, \quad . \quad (3)$$

the solution of Equations (2) is

$$x' : x'' : \cdots : x^{(n)} :: D' : D'' : \cdots D^{(n)}. \quad . \quad . \quad (4)$$

For example, the two equations

$$\left.\begin{array}{r} x + 4y - 3z = 0 \\ 5x - \ y - \ z = 0 \end{array}\right\}$$

give

$$x : y : z :: \begin{vmatrix} 4 & -3 \\ -1 & -1 \end{vmatrix} : - \begin{vmatrix} 1 & -3 \\ 5 & -1 \end{vmatrix} : \begin{vmatrix} 1 & 4 \\ 5 & -1 \end{vmatrix} :: 1 : 2 : 3.$$

45. Substituting in Equations (2) of the preceding article the values of the ratios

$$\frac{x'}{x^{(i)}},\ \frac{x''}{x^{(i)}},\ \cdots\ \frac{x^{(i-1)}}{x^{(i)}},\ \frac{x^{(i+1)}}{x^{(i)}},\ \cdots\ \frac{x^{(n)}}{x^{(i)}}$$

given by the proportions (4) of the same article, we obtain the $(n-1)$ equations

$$\left.\begin{array}{l} a_1'D' + a_1''D'' + \cdots + a_1^{(n)}D^{(n)} = 0, \\ a_2'D' + a_2''D'' + \cdots + a_2^{(n)}D^{(n)} = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n-1}'D' + a_{n-1}''D'' + \cdots + a_{n-1}^{(n)}D^{(n)} = 0. \end{array}\right\} \quad . . (1)$$

These $(n-1)$ relations are expressed by writing

$$\left\|\begin{matrix} a_1' & a_1'' & \cdots & a_1^{(n)} \\ a_2' & a_2'' & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1}' & a_{n-1}'' & \cdots & a_{n-1}^{(n)} \end{matrix}\right\| = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This expression is called a **rectangular array** or a **matrix.***

* The rotation $\begin{pmatrix} a_1' & a_1'' & \cdots & a_1^{(n)} \\ a_2' & a_2'' & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ a_n' & a_n'' & \cdots & a_n^{(n)} \end{pmatrix} = 0$

has of late years come into very general use. The discussion of the general theory of matrices is not within the scope of the present work.

In the rectangular array (2) the number of columns is one greater than the number of rows. In this case we have, by Equations (1),

$$a_k'\begin{vmatrix} a_1'' & a_1''' & \cdots & a_1^{(n)} \\ a_2'' & a_2''' & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1}'' & a_{n-1}''' & \cdots & a_{n-1}^{(n)} \end{vmatrix} - a_k''\begin{vmatrix} a_1' & a_1''' & \cdots & a_1^{(n)} \\ a_2' & a_2''' & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1}' & a_{n-1}''' & \cdots & a_{n-1}^{(n)} \end{vmatrix}$$

$$+\cdots+(-1)^{n-1}a_k^{(n)}\begin{vmatrix} a_1' & a_1'' & \cdots & a_1^{(n-1)} \\ a_2' & a_2'' & \cdots & a_2^{(n-1)} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1}' & a_{n-1}'' & \cdots & a_{n-1}^{(n-1)} \end{vmatrix}=0, \quad . . (3)$$

in which k may have any integral value from 1 to $(n-1)$ inclusive. Thus, in reference to the rectangular array

$$\left\|\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix}\right\|=0,$$

we have

$$a_1\,|\,b_1c_2d_3\,| - b_1\,|\,a_1c_2d_3\,| + c_1\,|\,a_1b_2d_3\,| - d_1\,|\,a_1b_2c_3\,| = 0,$$
$$a_2\,|\,b_1c_2d_3\,| - b_2\,|\,a_1c_2d_3\,| + c_2\,|\,a_1b_2d_3\,| - d_2\,|\,a_1b_2c_3\,| = 0,$$
$$a_3\,|\,b_1c_2d_3\,| - b_3\,|\,a_1c_2d_3\,| + c_3\,|\,a_1b_2d_3\,| - d_3\,|\,a_1b_2c_3\,| = 0.$$

46. Let us assume r homogeneous linear equations involving n unknowns, r being greater than n. They may be written as follows:

$$\left.\begin{array}{l} a_1'x' + a_1''x'' + \cdots + a_1^{(n)}x^{(n)} = 0, \\ a_2'x' + a_2''x'' + \cdots + a_2^{(n)}x^{(n)} = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'x' + a_n''x'' + \cdots + a_n^{(n)}x^{(n)} = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_r'x' + a_r''x'' + \cdots + a_r^{(n)}x^{(n)} = 0. \end{array}\right\} \quad . \quad . \quad . \quad (1)$$

If these equations are to be consistent, then, by Article 42, the determinant of each and every system of n equations taken from the above r equations must be equal to zero; that is, every determinant which may be formed from any n rows of the array

$$\begin{array}{ccc} a_1' & \cdots & a_1^{(n)} \\ a_2' & \cdots & a_2^{(n)} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ a_n' & \cdots & a_n^{(n)} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ a_r' & \cdots & a_r^{(n)} \end{array}$$

must be equal to zero.

The existence of this relation among the elements of this array is expressed by writing

$$\left\Vert \begin{array}{ccc} a_1' & \cdots & a_1^{(n)} \\ a_2' & \cdots & a_2^{(n)} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ a_n' & \cdots & a_n^{(n)} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ a_r' & \cdots & a_r^{(n)} \end{array} \right\Vert = 0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or, changing (arbitrarily) rows into columns and columns into rows, by

$$\left\|\begin{matrix} a_1' & a_2' & \cdots a_n' & \cdots a_r' \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{(n)} & a_2^{(n)} & \cdots a_n^{(n)} & \cdots a_r^{(n)} \end{matrix}\right\| = 0. \quad . \quad . \quad (3)$$

The condition of consistency for a system of r linear* equations involving n unknowns, r being greater than n, is thus concisely expressed in the form of a rectangular array. As an example, take the equations

$$\begin{aligned} 2x + y - z &= 0, \\ x - 2y + z &= 0, \\ x + 3y - 2z &= 0, \\ 4x - 3y + z &= 0. \end{aligned}$$

These are consistent because

$$\left.\begin{matrix} \begin{vmatrix} 2 & 1 & -1 \\ 1 & -2 & 1 \\ 1 & 3 & -2 \end{vmatrix} = 0, & \begin{vmatrix} 2 & 1 & -1 \\ 1 & -2 & 1 \\ 4 & -3 & 1 \end{vmatrix} = 0, \\ \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \\ 4 & -3 & 1 \end{vmatrix} = 0, & \begin{vmatrix} 1 & -2 & 1 \\ 1 & 3 & -2 \\ 4 & -3 & 1 \end{vmatrix} = 0. \end{matrix}\right\} \quad . \quad . \quad (4)$$

* If the equations are not homogeneous, then $a_1^{(n)}, a_2^{(n)}, \cdots a_n^{(n)}, \cdots a_r^{(n)}$ may represent the absolute terms, and the same conditions will still apply. (See Art. 41.)

These four relations are all expressed by writing

$$\left\|\begin{matrix} 2 & 1 & 1 & 4 \\ 1 & -2 & 3 & -3 \\ -1 & 1 & -2 & 1 \end{matrix}\right\| = 0.$$

By Formula (3) of Article 45 we should have

$$2\begin{vmatrix} 1 & 1 & 4 \\ -2 & 3 & -3 \\ 1 & -2 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 1 & 4 \\ 1 & 3 & -3 \\ -1 & -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 4 \\ 1 & -2 & -3 \\ -1 & 1 & 1 \end{vmatrix}$$

$$-4\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 3 \\ -1 & 1 & -2 \end{vmatrix} = 0,$$

with two other equations of a similar character, all of which may at once be seen to hold good on account of Equations (4).

EXAMPLES.

1. Test the consistency of the equations,

$$x + y + 2z = 9,$$
$$x + y - z = 0,$$
$$2x - y + z = 3,$$

and
$$x - 3y + 2z = 1.$$

2. If the equations

$$x - y - 2z = 0,$$
$$x - 2y + z = 0,$$

and
$$2x - 3y - z = 0$$

are consistent, find the ratios $x : y : z$.

3. Find the ratios of the unknowns in the equations

$$-4x + y + z = 0,$$

and

$$x - 2y + z = 0.$$

4. Find the ratios of the unknowns in the equations

$$2x + y - 2z = 0,$$
$$y + 4z - 4w = 0,$$

and

$$x - 5y + z + 2w = 0.$$

5. From the equations

$$\frac{ax + \gamma y + \beta z}{l} = \frac{\gamma x + by + \alpha z}{m} = \frac{\beta x + \alpha y + cz}{n},$$

deduce the relation

$$\frac{x}{\begin{vmatrix} \gamma & \beta & l \\ b & \alpha & m \\ \alpha & c & n \end{vmatrix}} = -\frac{y}{\begin{vmatrix} a & \beta & l \\ \gamma & \alpha & m \\ \beta & c & n \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & \gamma & l \\ \gamma & b & m \\ \beta & \alpha & n \end{vmatrix}}.$$

6. In the equations

$$a_1'x' + \cdots + a_1^{(n-1)}x^{(n-1)} + a_1^{(n)} = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_{n-1}'x' + \cdots + a_{n-1}^{(n-1)}x^{(n-1)} + a_{n-1}^{(n)} = 0,$$

show that

$$x' : x'' : \cdots : x^{(n-1)} : 1 :: D' : D'' : \cdots : D^{(n-1)} : D^{(n)}.$$

7. Solve the equations

$$2x + y + 3z = 19,$$
$$x - y + 2z = 7,$$

and

$$3x + 2y - z = 8,$$

by the formula given in the last example.

8. If any one of the equations of a system of n non-homogeneous linear equations (or of $n-1$ homogeneous linear equations) involving n unknowns is a consequence of the others, show that the equations fail to determine the values of the unknowns (or of their ratios).

9. Solve the equations

$$x + 3y - 2z = 1,$$

$$5x + 5y - 2z = 9,$$

and $$3x - y + 2z = 7.$$

47. The principles of the preceding articles may be applied to the solution of the problem of eliminating a single unknown from two consistent equations of any degree. We shall explain two methods of dealing with this important problem.

The simplest is Sylvester's dialytic method, which is as follows:

Let the two consistent equations from which it is required to eliminate the unknown be

$$\left.\begin{aligned} a_3x^3 + a_2x^2 + a_1x + a_0 = 0, \\ b_2x^2 + b_1x + b_0 = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

Multiplying the first of these equations by x, and the second by x and x^2 successively, we have the five consistent equations

$$\left.\begin{array}{rr} a_3x^3 + a_2x^2 + a_1x + a_0 = & 0, \\ a_3x^4 + a_2x^3 + a_1x^2 + a_0x \quad = & 0, \\ b_2x^2 + b_1x + b_0 = & 0, \\ b_2x^3 + b_1x^2 + b_0x \quad = & 0, \\ b_2x^4 + b_1x^3 + b_0x^2 \quad = & 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

These five Equations involve the four unknowns x^4, x^3, x^2, and x. Then, by Article 41, their eliminant is

$$\begin{vmatrix} 0 & a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & b_2 & b_1 & b_0 \\ 0 & b_2 & b_1 & b_0 & 0 \\ b_2 & b_1 & b_0 & 0 & 0 \end{vmatrix} = 0. \quad . \quad . \quad . \quad (3)$$

Equations (1) being consistent, Equation (3) is their eliminant; or, it being uncertain whether or not Equations (1) are consistent, Equation (3) is the condition which must be fulfilled in order that they may be so.

In general, let the two equations be

$$\left.\begin{array}{r} a_mx^m + a_{m-1}x^{m-1} + \cdots \quad \cdots \quad \cdots \quad \cdots + a_2x^2 + a_1x + a_0 = 0, \\ b_nx^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0 = 0. \end{array}\right\} \quad (4)$$

Multiplying the first equation by x, x^2, $\cdots$, and x^{n-1} successively, and the second by x, x^2, $\cdots$, and x^{m-1}, we have $(m+n)$ equations similar in form to Equations (2). They may be written as follows:

$$\left.\begin{array}{r}
a_m x^m + a_{m-1}x^{m-1} + \cdots \quad \cdots \quad \cdots \quad \cdots + a_1 x + a_0 = 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_m x^{m+n-1} + \cdots \quad \cdots \quad \cdots \quad \cdots + a_1 x^n + a_0 x^{n-1} \qquad = 0 \\
b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 = 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
b_m x^{m+n-1} + \cdots + b_1 x^m + b_0 x^{m-1} \qquad = 0
\end{array}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

These $(m+n)$ equations involve the $(m+n-1)$ unknowns, x^{m+n-1}, $\cdots$, x^2, and x. Then, by Article 41, their eliminant is

$$\left|\begin{array}{cccccccccccc}
0 & \cdots 0 & a_m & a_{m-1} \cdots & \cdots & \cdots & & \cdots & \cdots a_2 & a_1 & a_0 \\
0 & \cdots a_m & a_{m-1} & a_{m-2} \cdots & \cdots & \cdots & & \cdots & \cdots a_1 & a_0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_m \cdots & \cdots & \cdots & \cdots & \cdots a_2 & a_1 & a_0 & & \cdots 0 & 0 & 0 \\
0 & \cdots 0 & 0 & 0 & \cdots 0 & b_n & b_{n-1} \cdots & & b_2 & b_1 & b_0 \\
0 & \cdots 0 & 0 & 0 & \cdots b_n & b_{n-1} & b_{n-2} \cdots & & b_1 & b_0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
b_n & \cdots b_2 & b_1 & b_0 & \cdots 0 & 0 & 0 & & \cdots 0 & 0 & 0
\end{array}\right| = 0 \ldots (6)$$

This determinant is of the order $(m+n)$. It is of the nth degree in the coefficients of the equation of the mth degree, and of the mth degree in the coefficients of the equation of the nth.

The same method will apply when the two equations are homogeneous in two variables and it is required to eliminate either one or both variables.

As an example, let us eliminate x and y from the two equations

$$\begin{aligned} 2x^2y - xy^2 \qquad\qquad &= 0, \\ 8x^3y \qquad + 8xy^3 - 5y^4 &= 0. \end{aligned}$$

Dividing the first equation by y^3 and the second by y^4, they become

$$\begin{aligned} 2\frac{x^2}{y^2} - \frac{x}{y} \qquad\qquad &= 0, \\ 8\frac{x^3}{y^3} \qquad + 8\frac{x}{y} - 5 &= 0. \end{aligned}$$

Multiplying the first by $\frac{x}{y}$ and by $\frac{x^2}{y^2}$, and the second by $\frac{x}{y}$, we obtain, as from Equations (1), the eliminant

$$\begin{vmatrix} 0 & 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 & -5 \\ 8 & 0 & 8 & -5 & 0 \end{vmatrix} = 0;$$

a condition which, being fulfilled, shows that the given equations are consistent.

Sylvester's dialytic method may sometimes be applied in eliminating the unknowns from more than two equations. The following example, due to Professor Cayley, will illustrate.

Let the given equations be

$$x + y + z = 0,$$
$$x^2 = a,$$
$$y^2 = b,$$
and
$$z^2 = c.$$

Multiplying the first equation by xyz, by x, by y, and by z, we obtain, by the last three equations,

$$\begin{array}{rrrrl} & ayz + & bxz + & cxy & = 0, \\ a + & & xz + & xy & = 0, \\ b + & yz + & & xy & = 0, \\ c + & yz + & xz & & = 0, \end{array}$$

from which yz, xz, and xy may be eliminated as separate unknowns, giving

$$\begin{vmatrix} 0 & a & b & c \\ a & 0 & 1 & 1 \\ b & 1 & 0 & 1 \\ c & 1 & 1 & 0 \end{vmatrix} = 0.$$

Or, multiplying the first equation by yz, xz, and xy in succession, we may obtain the eliminant in the form

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c & b \\ 1 & c & 0 & a \\ 1 & b & a & 0 \end{vmatrix} = 0.$$

The eliminant of the equations

$$x+y+z=0,$$

$$x^3=a,$$

$$y^3=b,$$

and

$$z^3=c,$$

may be found in a similar manner.

48. We shall now consider Euler's method of elimination.

Let the given equations be

$$\left.\begin{aligned} F(x) &= a_3x^3+a_2x^2+a_1x+a_0=0, \\ f(x) &= \qquad\quad b_2x^2+b_1x+b_0=0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

The consistency of these equations requires that they have a common root, which we shall represent by r. Then

$$\left.\begin{aligned} \frac{F(x)}{x-r} &= \alpha_2x^2+\alpha_1x+\alpha_0, \\ \frac{f(x)}{x-r} &= \qquad\quad \beta_1x+\beta_0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which α_2, α_1, α_0, β_1, and β_0 are undetermined. Equations (2) give

$$(a_3x^3+a_2x^2+a_1x+a_0)(\beta_1x+\beta_0)$$
$$=(b_2x^2+b_1x+b_0)(\alpha_2x^2+\alpha_1x+\alpha_0).$$

Equating the coefficients of the like powers of x in the two members of this equation, we have

$$\left.\begin{array}{lllll} b_2\alpha_2 & & -a_3\beta_1 & & = 0, \\ b_1\alpha_2 + b_2\alpha_1 & & -a_2\beta_1 & -a_3\beta_0 & = 0, \\ b_0\alpha_2 + b_1\alpha_1 + b_2\alpha_0 & & -a_1\beta_1 & -a_2\beta_0 & = 0, \\ & b_0\alpha_1 + b_1\alpha_0 & -a_0\beta_1 & -a_1\beta_0 & = 0, \\ & b_0\alpha_0 & & -a_0\beta_0 & = 0. \end{array}\right\} \quad . \quad . \quad (3)$$

Eliminating α_2, α_1, α_0, β_1, and β_0 from the above system gives

$$\begin{vmatrix} b_2 & 0 & 0 & a_3 & 0 \\ b_1 & b_2 & 0 & a_2 & a_3 \\ b_0 & b_1 & b_2 & a_1 & a_2 \\ 0 & b_0 & b_1 & a_0 & a_1 \\ 0 & 0 & b_0 & 0 & a_0 \end{vmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is the eliminant of Equations (1).

This determinant is the same as the determinant (3) of Article 47.

In general, let the given equations be

$$F(x) = a_m x^m + \cdots + a_2 x^2 + a_1 x + a_0 = 0,$$

$$f(x) = b_n x^n + \cdots + b_2 x^2 + b_1 x + b_0 = 0.$$

Then, r being a common root of these equations, we have

$$\frac{F(x)}{x-r} = \alpha_{m-1}x^{m-1} + \cdots + \alpha_1 x + \alpha_0 = \Phi(x),$$

$$\frac{f(x)}{x-r} = \beta_{n-1}x^{n-1} + \cdots + \beta_1 x + \beta_0 = \phi(x),$$

in which $\alpha_{m-1}, \ldots, \alpha_1, \alpha_0, \beta_{n-1}, \ldots, \beta_1$, and β_0 are $(m+n)$ undetermined quantities. These equations give

$$F(x) \cdot \phi(x) = f(x) \cdot \Phi(x),$$

each member of which is of the degree $(m+n-1)$ in x, and by equating coefficients we may obtain $(m+n)$ equations which are linear and homogeneous with respect to the $(m+n)$ undetermined quantities $\alpha_{m-1}, \ldots \alpha_0, \beta_{n-1}, \ldots \beta_0$. The determinant of the coefficients of these $(m+n)$ equations is the required eliminant. This determinant is the same as the determinant (6) of Article 47 except that columns of the one appear as rows in the other, and *vice versa*.

By means of Euler's method we may readily find the conditions which must be fulfilled in order that two equations may have two or more roots in common. For example, let the equations be

$$F(x) = a_3x^3 + a_2x^2 + a_1x + a_0 = 0,$$

$$f(x) = b_3x^3 + b_2x^2 + b_1x + b_0 = 0,$$

the common roots being r_1 and r_2.

Then, as before,

$$\frac{F(x)}{(x-r_1)(x-r_2)} = \alpha_1 x + \alpha_0,$$

$$\frac{f(x)}{(x-r_1)(x-r_2)} = \beta_1 x + \beta_0.$$

These equations give

$$(a_3x^3 + a_2x^2 + a_1x + a_0)(\beta_1 x + \beta_0)$$
$$= (b_3x^3 + b_2x^2 + b_1x + b_0)(\alpha_1 x + \alpha_0).$$

Equating the coefficients of the like powers of x in the two members of this equation, we obtain

$$\begin{array}{l}
a_3\beta_1 \qquad\qquad - b_3\alpha_1 \qquad\qquad = 0,\\
a_2\beta_1 + a_3\beta_0 - b_2\alpha_1 - b_3\alpha_0 = 0,\\
a_1\beta_1 + a_2\beta_0 - b_1\alpha_1 - b_2\alpha_0 = 0,\\
a_0\beta_1 + a_1\beta_0 - b_0\alpha_1 - b_1\alpha_0 = 0,\\
\qquad\quad a_0\beta_0 - \qquad\qquad b_0\alpha_0 = 0.
\end{array}$$

These five equations, being homogeneous and linear with respect to the four undetermined quantities β_1, β_0, α_1, and α_0, the conditions of their consistency, that is, the conditions which must be fulfilled in order that the given equations may have two roots in common, are (Art. 46) expressed by the rectangular array

$$\left\|\begin{array}{ccccc}
0 & a_0 & a_1 & a_2 & a_3\\
a_0 & a_1 & a_2 & a_3 & 0\\
0 & b_0 & b_1 & b_2 & b_3\\
b_0 & b_1 & b_2 & b_3 & 0
\end{array}\right\| = 0.$$

The student can readily generalize this process for himself.

EXAMPLES.

1. Test the consistency of the equations

$$2x^3 - 8x^2 + 6x = 0,$$

and

$$x^2 - 2x - 3 = 0,$$

by Sylvester's method, and also by that of Euler.

2. Test the consistency of the equations

$$6x^3 - 3x^2y - xy^2 - 12y^3 = 0,$$

and

$$4x^2 - 8xy + 3y^2 = 0.$$

3. Eliminate x from the equations in the last example, thus obtaining an equation in y only.

4. Eliminate x and y from the equations

$$ax^2z + bxy + c = 0,$$

$$dyz + ex = 0,$$

and

$$fxz + gx + h = 0,$$

thus obtaining a single equation in z.

Ans.
$$\begin{vmatrix} az & b & 0 & 0 & c \\ e & dz & 0 & 0 & 0 \\ (fz+g) & 0 & h & 0 & 0 \\ 0 & (fz+g) & 0 & h & 0 \\ 0 & 0 & (fz+g) & 0 & h \end{vmatrix} = 0.$$

5. Free from radicals the equation

$$\sqrt{a_1x+a_0}+\sqrt{b_1x+b_0}+c_0=0.$$

SOLUTION. — Let

$$a_1x+a_0=y^2, \qquad (1)$$

and

$$b_1x+b_0=z^2. \qquad (2)$$

The given equation thus becomes

$$y+z+c_0=0. \qquad (3)$$

Eliminating z from (2) and (3) gives

$$\begin{vmatrix} 1 & 0 & -(b_1x+b_0) \\ 0 & 1 & (y+c_0) \\ 1 & (y+c_0) & 0 \end{vmatrix}=0,$$

or

$$y^2+2\,c_0y-(b_1x+b_0-c_0^2)=0. \qquad (4)$$

Eliminating y from (1) and (4) gives

$$\begin{vmatrix} 0 & 1 & 0 & -(a_1x+a_0) \\ 1 & 0 & -(a_1x+a_0) & 0 \\ 0 & 1 & 2\,c_0 & -(b_1x+b_0-c_0^2) \\ 1 & 2\,c_0 & -(b_1x+b_0-c_0^2) & 0 \end{vmatrix}=0,$$

which is an equation in x and free from radicals.

6. Eliminate the unknowns from the equations

$$x+y+z=0,$$
$$x^3=a,$$
$$y^3=b,$$
$$z^3=c.$$

Ans. One form for the eliminant is

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & c & b & 0 & 0 & 0 & 1 & 0 & 0 \\ c & 0 & a & 0 & 0 & 0 & 0 & 1 & 0 \\ b & a & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & b & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & c & 1 & 1 & 0 \end{vmatrix} = 0.$$

7. Find the condition that all the roots of the equation

$$x^3 + 3Hx + G = 0$$

shall be real.

Solution. — The three roots may be represented by

$$\alpha,\ \beta + \sqrt{\gamma^2},\ \text{and}\ \beta - \sqrt{\gamma^2}.$$

These will all be real when

$$\gamma^2 > 0, \quad \ldots \quad (1)$$

and the last two will be imaginary when

$$\gamma^2 < 0 \quad \ldots \quad (2)$$

From the well-known relations between the roots and coefficients of the rational integral function, we have

$$\left.\begin{aligned} \alpha + 2\beta &= 0, \\ 2\alpha\beta + \beta^2 - \gamma^2 &= 3H, \\ \alpha\beta^2 - \alpha\gamma^2 &= -G. \end{aligned}\right\} \quad \ldots \quad (3)$$

Eliminating α and β from Equations (3) gives

$$\begin{vmatrix} 0 & 0 & 3 & 0 & (\gamma^2+3H) \\ 0 & 3 & 0 & (\gamma^2+3H) & 0 \\ 3 & 0 & (\gamma^2+3H) & 0 & 0 \\ 0 & 2 & 0 & -2\gamma^2 & -G \\ 2 & 0 & -2\gamma^2 & -G & 0 \end{vmatrix} = 0.$$

This readily reduces to

$$\gamma^2 = -\frac{27}{4}\frac{G^2+4H^3}{(4\gamma^2+9H)^2},$$

which, compared with (1), shows that the roots are all real when

$$G^2+4H^3<0,$$

the required condition. When

$$G^2+4H^3>0,$$

the two conjugate roots are imaginary.

The function G^2+4H^3 is called the *discriminant* of the cubic $x^3+3Hx+G=0$.

8. Find the condition to be fulfilled in order that two of the roots of the equation

$$x^3+3ax^2+3bx+c=0$$

may be equal.

SOLUTION. — Let α, α, and β be the three roots, and we readily obtain

$$\begin{aligned} 2\alpha+\beta &= -3a, \\ \alpha^2+2\alpha\beta &= 3b, \\ \alpha^2\beta &= -c. \end{aligned}$$

From the first of these equations

$$\beta = -(3a+2\alpha),$$

which reduces the second and third to

$$\begin{aligned} \alpha^2 + 2\,a\alpha + b &= 0, \\ 2\,\alpha^3 + 3\,a\alpha^2 \qquad\qquad - c &= 0, \end{aligned}$$

respectively.

Eliminating α from these last equations by Sylvester's method gives

$$\begin{vmatrix} 0 & 0 & 1 & 2\,a & b \\ 0 & 1 & 2\,a & b & 0 \\ 1 & 2\,a & b & 0 & 0 \\ 0 & 2 & 3\,a & 0 & -c \\ 2 & 3\,a & 0 & -c & 0 \end{vmatrix} = 0,$$

or,
$$4\,a^3c - 3\,a^2b^2 - 6\,abc + 4\,b^3 + c^2 = 0, \quad \cdots \quad (1)$$

the required condition.

Or, since a double root is common to the given equation and its first derivative, we may apply Euler's method of elimination to the two equations

$$\left.\begin{aligned} x^3 + 3\,ax^2 + 3\,bx + c = 0, \\ x^2 + 2\,ax + b = 0. \end{aligned}\right\} \quad \cdots \quad (2)$$

We should thus obtain the same condition as before.

9. Find the double root of the equation

$$x^3 + 3\,ax^2 + 3\,bx + c = 0,$$

when the condition expressed by Equation (1) of the last example is fulfilled.

SOLUTION. — Equations (2) in the last example give

$$\begin{aligned} (x + 3\,a)\,x^2 + 3\,bx + c &= 0, \\ (x + 2\,a)\,x^2 + bx \phantom{{}+c} &= 0, \\ x^2 + 2\,ax + b &= 0. \end{aligned}$$

Eliminating x^2 and x from these equations, we obtain

$$\begin{vmatrix} (x+3a) & 3b & c \\ (x+2a) & b & 0 \\ 1 & 2a & b \end{vmatrix} = 0,$$

the solution of which furnishes the double root. The required value is

$$x = \frac{3a(b^2 - ac) + c(b - a^2)}{2(b^2 - ac)}.$$

10. Find the condition that the equation

$$x^3 + 3Hx + G = 0$$

shall have a double root, and find the value of such a root when the condition is fulfilled.

Ans. $G^2 + 4H^3 = 0;\ -\dfrac{G}{2H}.$

In the equation

$$x^3 + 3ax^2 + 3bx + c = 0,$$

find the condition:

11. That the three roots may be in arithmetical progression. *Ans.* $6a^3 - 9ab + 3c = 0.$

12. That the three roots may be in harmonical progression. *Ans.* $6b^3 - 9abc + 3c^2 = 0.$

13. That the three roots may be in geometrical progression. *Ans.* $9a^3b^3 - c(a^2 - ab + b^2)^2 = 0.$

14. That one root may be double another.

Ans. $18(54a^3c - 36a^2b^2 - 91abc + 54b^3) + 343c^2 = 0.$

15. That the three roots may have the ratios 1 : 2 : 3.

Ans. $11c - 9ab = 0.$

49. Let it be required to determine whether the two linear factors of a homogeneous quadratic function of two variables are real and unequal, real and equal, or imaginary.

Every such function may be written in the form

$$ax^2 + by^2 + 2\,hxy.$$

Placing this equal to zero and solving for x gives

$$x = -\frac{1}{a}\left[h \pm (h^2 - ab)^{\frac{1}{2}}\right] y,$$

which shows that the factors are real and unequal when

$$h^2 - ab > 0,$$

real and equal when

$$h^2 - ab = 0,$$

and imaginary when

$$h^2 - ab < 0.$$

The function $(h^2 - ab)$ may be expressed as a determinant thus,

$$-(h^2 - ab) = \begin{vmatrix} a & h \\ h & b \end{vmatrix} \equiv D.$$

The determinant D is called the **discriminant** of the given quadratic function, $ax^2 + by^2 + 2\,hxy$.

The required conditions may now be stated as follows:

The linear factors of a homogeneous quadratic function of two variables are real and unequal, real and equal, or imaginary, according as the discrimi-

nant of the function is less than, equal to, or greater than zero, respectively.

The same conditions determine the nature of the binomial factors of the complete quadratic function of a single variable, viz.:

$$ax^2 + 2hx + b,$$

as may readily be seen by making $y = 1$ in the preceding case.

50. Let us now determine the condition which must be fulfilled in order that a homogeneous quadratic function of three variables may be resolved into two rational linear factors.

Every such function may be written in the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy.$$

Placing this equal to zero and solving for x gives

$$x = -\frac{1}{a}\{(hy + gz) \pm [(h^2 - ab)y^2 + (g^2 - ac)z^2 + 2(hg - af)yz]^{\frac{1}{2}}\}. \quad . \quad . \quad . \quad . \quad (1)$$

If the quadratic function under the radical in the second member of this equation is a perfect square, the given function can be resolved into two rational linear factors. By the preceding article, the condition that the quantity under the radical may be a perfect square is

$$\begin{vmatrix} (h^2-ab) & (hg-af) \\ (hg-af) & (g^2-ac) \end{vmatrix} = 0. \quad . \quad . \quad . \quad (2)$$

This equation expresses the required condition. The determinant in the left-hand member may however be transformed into a simpler and more symmetrical form. Thus:

$$\begin{vmatrix} (h^2-ab) & (hg-af) \\ (hg-af) & (g^2-ac) \end{vmatrix} = \frac{1}{a}\begin{vmatrix} a & h & g \\ 0 & (ab-h^2) & (af-hg) \\ 0 & (af-hg) & (ac-g^2) \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} a & h & g \\ ah & ab & af \\ ag & af & ac \end{vmatrix} = a\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0. \quad . \quad . \quad (3)$$

In the same manner we may deduce for the required condition either

$$b\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, \text{ or } c\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0. \quad . \quad (4), (5)$$

In order that one and all of the conditions (3), (4), and (5) may be fulfilled we must have

$$D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, \quad . \quad . \quad . \quad (6)$$

or we must have

$$a = 0, \quad b = 0, \quad \text{and} \quad c = 0,$$

simultaneously.

This last supposition reduces the given quadratic function to

$$fxz + gyz + hxy,$$

which is a prime function of the three variables unless one of the coefficients f, g, or h vanishes along with a, b, and c; but in this case the determinant D must of necessity vanish.

Hence Equation (6) in every case expresses the required condition.

The determinant D in this article is called the **discriminant** of the quadratic function

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy.$$

We may now state the result obtained above in the following form:

If the discriminant of a homogeneous quadratic function of three variables vanishes, the function may be resolved into two linear factors.

The same condition determines whether or not the complete quadratic function of two variables, viz.:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c,$$

may be resolved into two linear factors, as may readily be seen by making $z=1$ in the case just considered.

The student familiar with the calculus may have noticed that the discriminant is the eliminant of the first derivatives of the function to which it pertains, taken with respect to each of its variables.

It is a function of great importance in the theory of homogeneous functions or *quantics*.

EXAMPLES.

Tell whether the roots of the following quadratic equations are real and unequal, real and equal, or imaginary.

1. $2x^2-5x-9=0$.
2. $x^2-7x+3=0$.
3. $3x^2+x+3=0$.
4. $\frac{1}{x^2}-\frac{5}{x}+9=0$.
5. $4x^2-3x=0$.
6. $\frac{x^2}{2}+x+\frac{1}{2}=0$.
7. $(x+3)^2+(x-3)^2=0$.
8. $2x^2-\frac{x}{5}+10=0$.
9. $lx^2-mx+\frac{1+m^2}{l}=0$.

Tell whether or not the following functions are prime.

10. $x^2+z^2+yz+2xz+xy$.

11. $5x^2 - 9y^2 - 5z^2 + 18yz + 24xz - 12xy.$

12. $ax^2 - bxy + cxz.$

13. $x^2 - y^2 + 2y - 1.$

14. $9y^2 + 15yz - 6xz + 8xy.$

15. $ll'x^2 + mm'y^2 + nn'z^2 + (mn' + m'n)yz$
$$+ (ln' + l'n)xz + (lm' + l'm)xy.$$

Find the values of λ in order that each of the following functions may be resolved into linear factors.

16. $12x^2 + \lambda xy + 2y^2 + 11x - 5y + 2.$

17. $\lambda xy + 5x + 3y + 2.$

18. $2x^2 - 3\lambda y^2 - 12z^2 + 17yz + \lambda xz - xy.$

If A, B, C, F, G, H be the co-factors of a, b, c, f, g, h respectively in the discriminant D, then show that

19. $\begin{vmatrix} B & F \\ F & C \end{vmatrix} = 0$ 20. $\begin{vmatrix} A & G \\ G & C \end{vmatrix} = 0.$ 21. $\begin{vmatrix} A & H \\ H & B \end{vmatrix} = 0.$

22. $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0.$ (See Art. 60.)

23. Show that, in order that

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy$$

may be a perfect square, we must have

$$\begin{vmatrix} b & f \\ f & c \end{vmatrix} = 0, \qquad \begin{vmatrix} a & g \\ g & c \end{vmatrix} = 0, \qquad \begin{vmatrix} a & h \\ h & b \end{vmatrix} = 0.$$

24. By means of the preceding examples show that when the function

$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy$$

breaks up into two linear factors, then is the function

$$Au^2+Bv^2+Cw^2+2Fvw+2Guw+2Huv$$

a perfect square.

25. If $(lx+my+nz)$ and $(l'x+m'y+n'z)$ are the factors of

$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy,$$

we have, by equating coefficients

$$ll'=a, \qquad mm'=b, \qquad nn'=c,$$
$$mn'+m'n=2f, \quad ln'+l'n=2g, \quad lm'+l'm=2h.$$

Hence, by eliminating l, m, n, l', m', n', from these equations, obtain the discriminant in the form

$$D\equiv abc+2fgh-af^2-bg^2-ch^2=0.$$

CHAPTER VI.

MULTIPLICATION OF DETERMINANTS, AND RECIPROCAL DETERMINANTS.

The process of multiplication in determinants and some of the more important of the many interesting results to which it leads are considered in the present chapter.

51. Let us assume the two simultaneous equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 = 0, \\ a_{21}x_1 + a_{22}x_2 = 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which

$$\left.\begin{aligned} x_1 = b_{11}u_1 + b_{21}u_2, \\ x_2 = b_{12}u_1 + b_{22}u_2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting these values of x_1 and x_2 in Equations (1), we have

$$\left.\begin{aligned} (a_{11}b_{11} + a_{12}b_{12})u_1 + (a_{11}b_{21} + a_{12}b_{22})u_2 = 0, \\ (a_{21}b_{11} + a_{22}b_{12})u_1 + (a_{21}b_{21} + a_{22}b_{22})u_2 = 0. \end{aligned}\right\} \quad . \quad . \quad (3)$$

These last equations are simultaneously satisfied for values of u_1 and u_2 other than zero only when (Art. 42)

$$\begin{vmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{21} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22} \end{vmatrix} = 0. \quad . \quad (4)$$

Now Equations (3) will be satisfied whenever Equations (1) are satisfied, and Equations (1) are satisfied either when

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or when

$$x_1 = x_2 = 0.$$

But, on account of Equations (2), this last condition requires that

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence, Equation (4) holds good when either (5) or (6) holds good. It follows that the determinants (5) and (6) are factors of the determinant (4), and if this contains any other factor it must appear in every term of the expansion. One term of the expansion of (4) is $a_{11}b_{11}a_{22}b_{22}$; but this is also a term in the product of the expansions of the determinants in (5) and (6).

Therefore

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{21} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22} \end{vmatrix}. \quad (7)$$

It may be shown in a precisely similar manner that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11}b_{11}+a_{12}b_{12}+a_{13}b_{13} & a_{11}b_{21}+a_{12}b_{22}+a_{13}b_{23} & a_{11}b_{31}+a_{12}b_{32}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{12}+a_{23}b_{13} & a_{21}b_{21}+a_{22}b_{22}+a_{23}b_{23} & a_{21}b_{31}+a_{22}b_{32}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{12}+a_{33}b_{13} & a_{31}b_{21}+a_{32}b_{22}+a_{33}b_{23} & a_{31}b_{31}+a_{32}b_{32}+a_{33}b_{33} \end{vmatrix} . \quad \ldots \quad (8)$$

52. The same method may be applied to the formation of the product of any two determinants of the same order. An inspection of the two products already written enables us to write the following general rule:

To form the determinant $|p_{1n}|$, *which is the product of two determinants* $|a_{1n}|$ *and* $|b_{1n}|$, *first connect by plus signs the elements in the rows of both* $|a_{1n}|$ *and* $|b_{1n}|$. *Now place the first row of* $|a_{1n}|$ *upon each row of* $|b_{1n}|$ *in turn, and let each two elements as they touch become products. This is the first row of* $|p_{1n}|$. *Perform the same operation upon* $|b_{1n}|$ *with the second row of* $|a_{1n}|$ *to obtain the second row of* $|p_{1n}|$; *and again with the third row of* $|a_{1n}|$ *to obtain the third row of* $|p_{1n}|$; *etc.**

When it is required to form the product of two determinants of different orders, the one of lower order may be expressed as a determinant of the same order as the other. (See Art. 30.)

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* See Carr's *Synopsis of Pure Mathematics*, Article 570.

Their product may then be formed by the usual method.

53. In accordance with the above rule we have the following general formula:

$$\begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \cdots b_{1n} \\ b_{21} & b_{22} \cdots b_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \\ b_{n1} & b_{n2} \cdots b_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11}b_{11}+\cdots+a_{1n}b_{1n} & a_{11}b_{21}+\cdots+a_{1n}b_{2n} & \cdots & a_{11}b_{n1}+\cdots+a_{1n}b_{nn} \\ a_{21}b_{11}+\cdots+a_{2n}b_{1n} & a_{21}b_{21}+\cdots+a_{2n}b_{2n} & \cdots & a_{21}b_{n1}+\cdots+a_{2n}b_{nn} \\ \cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot & & \cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{n1}b_{11}+\cdots+a_{nn}b_{1n} & a_{n1}b_{21}+\cdots+a_{nn}b_{2n} & \cdots & a_{n1}b_{n1}+\cdots+a_{nn}b_{nn} \end{vmatrix} \quad . \quad . \quad . \quad . \quad (1)$$

If this product be represented by $|p_{1n}|$, the element in the kth row and the sth column will be

$$p_{ks} = a_{k1}b_{s1} + a_{k2}b_{s2} + \cdots + a_{kn}b_{sn}. \quad . \quad . \quad . \quad (2)$$

NOTE.—Instead of forming the product of two determinants by rows, as has been done above, the product may be formed by columns. The student will find no difficulty in writing out the corresponding formulæ for himself.

54. Given the system of equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0. \end{array}\right\} \quad . \quad . \quad . \quad (1)$$

Let it be required to transform the above system into one in the variables u_1, u_2, $\cdots$ u_n by means of the following linear substitutions: *

$$\left.\begin{array}{l} x_1 = b_{11}u_1 + b_{21}u_2 + \cdots + b_{n1}u_n, \\ x_2 = b_{12}u_1 + b_{22}u_2 + \cdots + b_{n2}u_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n = b_{1\,n}u_1 + b_{2\,n}u_2 + \cdots + b_{nn}u_n. \end{array}\right\} \quad . \quad . \quad . \quad (2)$$

The determinant

$$\begin{vmatrix} b_{11} & b_{12} \cdots b_{1n} \\ b_{21} & b_{22} \cdots b_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \\ b_{n1} & b_{n2} \cdots b_{nn} \end{vmatrix},$$

formed from the coefficients of Equations (2), is called the *modulus of transformation.* When this modulus is unity, the transformation is said to be *uni-modular.*

By what precedes in this chapter it will readily be seen that, if the transformed equations be written

$$\left.\begin{array}{l} p_{11}u_1 + p_{12}u_2 + \cdots + p_{1n}u_n = 0, \\ p_{21}u_1 + p_{22}u_2 + \cdots + p_{2n}u_n = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p_{n1}u_1 + p_{n2}u_2 + \cdots + p_{nn}u_n = 0, \end{array}\right\} \quad . \quad . \quad . \quad (3)$$

---

* The transformation from one set of rectangular axis to another in Analytical Geometry is of this sort. Hence the great importance of linear substitutions. (See Chap. IX.)

then will

$$\begin{vmatrix} p_{11} & p_{12} \cdots p_{1n} \\ p_{21} & p_{22} \cdots p_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \\ p_{n1} & p_{n2} \cdots p_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \cdots b_{1n} \\ b_{21} & b_{22} \cdots b_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \\ b_{n1} & b_{n2} \cdots b_{nn} \end{vmatrix};$$

that is: *If a system of* n *homogeneous linear equations in* n *variables be subjected to linear transformation, the determinant of the coefficients of the transformed equations will be the determinant of the coefficients of the given equations multiplied by the modulus of transformation.*

55. It has been shown by Professor Sylvester that the product of two determinants of the nth order may, by bordering the determinants as explained in Article 30 before forming the product, be represented in $(n+1)$ distinct forms. For the case in which $n=3$, we have in addition to the form written in Article 51, the following:

Writing the determinants in the forms (see Ex. 1, after Art. 30)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad \text{and} - \begin{vmatrix} b_{11} & b_{12} & 0 & b_{13} \\ b_{21} & b_{22} & 0 & b_{23} \\ b_{31} & b_{32} & 0 & b_{33} \\ 0 & 0 & 1 & 0 \end{vmatrix},$$

their product is the determinant of the fourth order

$$-\begin{vmatrix} a_{11}b_{11}+a_{12}b_{12} & a_{11}b_{21}+a_{12}b_{22} & a_{11}b_{31}+a_{12}b_{32} & a_{13} \\ a_{21}b_{11}+a_{22}b_{12} & a_{21}b_{21}+a_{22}b_{22} & a_{21}b_{31}+a_{22}b_{32} & a_{23} \\ a_{31}b_{11}+a_{32}b_{12} & a_{31}b_{21}+a_{32}b_{22} & a_{31}b_{31}+a_{32}b_{32} & a_{33} \\ b_{13} & b_{23} & b_{33} & 0 \end{vmatrix}.$$

Again, writing the given determinants in the forms

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} b_{11} & 0 & 0 & b_{12} & b_{13} \\ b_{21} & 0 & 0 & b_{22} & b_{23} \\ b_{31} & 0 & 0 & b_{32} & b_{33} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix},$$

their product is the determinant of the fifth order,

$$\begin{vmatrix} a_{11}b_{11} & a_{11}b_{21} & a_{11}b_{31} & a_{12} & a_{13} \\ a_{21}b_{11} & a_{21}b_{21} & a_{21}b_{31} & a_{22} & a_{23} \\ a_{31}b_{11} & a_{31}b_{21} & a_{31}b_{31} & a_{32} & a_{33} \\ b_{12} & b_{22} & b_{32} & 0 & 0 \\ b_{13} & b_{23} & b_{33} & 0 & 0 \end{vmatrix}.$$

Once more, writing the determinants in the forms

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{vmatrix},$$

their product is the determinant of the sixth order,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 0 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 & b_{31} & b_{32} & b_{33} \end{vmatrix}.$$

By this method the product of two determinants of the nth order may be expressed as a determinant of any order from the nth to the $2n$th inclusive.

56. Formula (1) of Article 53 may be used in forming what may, conventionally, be called the product of two rectangular arrays of equal dimensions, such as

$$\left.\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}\right\} \quad \text{and} \quad \left.\begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{matrix}\right\}.$$

Representing the result of the operation by Δ, we have

$$\Delta \equiv \begin{vmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{21} + a_{12}b_{22} & a_{11}b_{31} + a_{12}b_{32} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22} & a_{21}b_{31} + a_{22}b_{32} \\ a_{31}b_{11} + a_{32}b_{12} & a_{31}b_{21} + a_{32}b_{22} & a_{31}b_{31} + a_{32}b_{32} \end{vmatrix} = 0.$$

This determinant is equal to zero, because it may be regarded as the product of the two determinants

which we should obtain by adding a column of zeros to one of the given rectangular arrays and a column of elements arbitrarily chosen to the other; thus:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & \beta_{13} \\ b_{21} & b_{22} & \beta_{23} \\ b_{31} & b_{32} & \beta_{33} \end{vmatrix} = 0.$$

In a similar manner it may be shown that

The product (so-called) of any two rectangular arrays having m *columns and* n *rows, in which* $m < n$, *is a determinant whose value is zero.*

57. If the two rectangular arrays are

$$\left.\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix}\right\} \text{ and } \left.\begin{matrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{matrix}\right\},$$

the operation of multiplication gives

$$\Delta \equiv \begin{vmatrix} a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} & a_{11}b_{21} + a_{12}b_{22} + a_{13}b_{23} \\ a_{21}b_{11} + a_{22}b_{12} + a_{23}b_{13} & a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} \end{vmatrix},$$

which may be written in the form

$$\Delta = \begin{vmatrix} a_{12}b_{12} + a_{13}b_{13} & a_{12}b_{22} + a_{13}b_{23} \\ a_{22}b_{12} + a_{23}b_{13} & a_{22}b_{22} + a_{23}b_{23} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11}b_{11} + a_{13}b_{13} & a_{11}b_{21} + a_{13}b_{23} \\ a_{21}b_{11} + a_{23}b_{13} & a_{21}b_{21} + a_{23}b_{23} \end{vmatrix}$$

$$+\begin{vmatrix} a_{11}b_{11}+a_{12}b_{12} & a_{11}b_{21}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{12} & a_{21}b_{21}+a_{22}b_{22} \end{vmatrix} - \begin{vmatrix} a_{13}b_{13} & a_{13}b_{23} \\ a_{23}b_{13} & a_{23}b_{23} \end{vmatrix}$$

$$-\begin{vmatrix} a_{12}b_{12} & a_{12}b_{22} \\ a_{22}b_{12} & a_{22}b_{22} \end{vmatrix} - \begin{vmatrix} a_{11}b_{11} & a_{11}b_{21} \\ a_{21}b_{11} & a_{21}b_{21} \end{vmatrix}.$$

Each of the first three of these determinants is the product of two determinants, and each of the last three is equal to zero. Hence

$$\Delta = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} b_{12} & b_{13} \\ b_{22} & b_{23} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{13} \\ b_{21} & b_{23} \end{vmatrix}$$

$$+\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}.$$

This may be generalized as follows:

The product (so-called) of any two rectangular arrays having m *columns and* n *rows, in which* $m > n$, *is equal to the sum of the* $\dfrac{m(m-1)\cdots(m-n+1)}{n(n-1)\cdots 1}$ *determinants which may be formed from one of the arrays by deleting* $(m-n)$ *columns, each multiplied by the corresponding determinant formed from the other array.*

58. We shall now demonstrate the following theorem, due to Leonard Euler:

The product of two numbers, each the sum of four squares, is itself the sum of four squares.

From Example 8, after Article 38, we have

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2. \quad (1)$$

Similarly

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ -\beta & \alpha & -\delta & \gamma \\ -\gamma & \delta & \alpha & -\beta \\ -\delta & -\gamma & \beta & \alpha \end{vmatrix} = (\alpha^2+\beta^2+\gamma^2+\delta^2)^2 \quad (2)$$

Let Equations (1) and (2) be multiplied together, member for member.

Letting

$$\begin{aligned} a\alpha + b\beta + c\gamma + d\delta &= A, \\ -a\beta + b\alpha - c\delta + d\gamma &= B, \\ -a\gamma + b\delta + c\alpha - d\beta &= C, \\ -a\delta - b\gamma + c\beta + d\alpha &= D, \end{aligned}$$

the product of the left-hand members may be written

$$\begin{vmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{vmatrix} = (A^2+B^2+C^2+D^2)^2. \quad (3)$$

Hence

$$(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)$$
$$=(A^2+B^2+C^2+D^2),$$

which is the theorem.

It may be shown that the product of two numbers, each the sum of four squares, may be expressed as the sum of four squares in forty-eight (48) different ways. The product of n numbers, each the sum of four squares, may be expressed as the sum of four squares in $(48)^{n-1}$ different ways.

EXAMPLES.

1. Show that the product of any number of determinants, of the same or different orders, may be obtained as a determinant of the order which is highest among the factors.

Perform the following indicated multiplications:

2. $$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

3. $$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} \cdot \begin{vmatrix} a+\lambda & h & g \\ h & b+\lambda & f \\ g & f & c+\lambda \end{vmatrix}.$$

4. $$\begin{vmatrix} 0 & \beta & \gamma \\ \alpha & 0 & \gamma \\ \alpha & \beta & 0 \end{vmatrix} \cdot \begin{vmatrix} \beta & \gamma & 0 \\ \beta & 0 & \alpha \\ 0 & \gamma & \alpha \end{vmatrix}.$$

5. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \cdot \begin{vmatrix} a & g \\ g & c \end{vmatrix}.$

6. $\begin{vmatrix} b & f \\ f & c \end{vmatrix} \cdot \begin{vmatrix} a & g \\ g & c \end{vmatrix} \cdot \begin{vmatrix} a & h \\ h & b \end{vmatrix}.$

7. Form the product

$$\begin{vmatrix} x_2+x_3 & x_1+x_3 & x_1+x_2 \\ y_2+y_3 & y_1+y_3 & y_1+y_2 \\ z_2+z_3 & z_1+z_3 & z_1+z_2 \end{vmatrix} \cdot \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix},$$

and thence show that the first of the two determinants is equal to

$$2\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

8. Obtain the quotient,

$$\begin{vmatrix} -ma+nb & mb+nc & -mc+na \\ la+nb & -lb+nc & lc+na \\ la-ma & -lb+mb & lc-mc \end{vmatrix} \div \begin{vmatrix} 0 & m & n \\ l & 0 & n \\ l & m & 0 \end{vmatrix},$$

in the form of a determinant, and thence show that the first determinant equals zero.

9. Find two determinants whose product is

$$\begin{vmatrix} ax+cx & al+cn & ay+cy \\ bm & 0 & bm \\ cx+ax & cl+an & cy+ay \end{vmatrix}.$$

10. Prove the equality:

$$\begin{vmatrix} a_1^2+b_1+c_1 & a_1a_2+b_2+c_2 & a_1a_3+b_3+c_3 \\ b_2b_1+c_1 & b_2^2+c_2 & b_2b_3+c_3 \\ c_3c_1 & c_3c_2 & c_3^2 \end{vmatrix}$$

$$= a_1b_2c_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(See Eq. 3, Art. 29.)

11. Show by means of the multiplication theorem that

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

$$= -\tfrac{1}{3}\begin{vmatrix} a_1+b_1+c_1 & a_1+b_1+d_1 & a_1+c_1+d_1 & b_1+c_1+d_1 \\ a_2+b_2+c_2 & a_2+b_2+d_2 & a_2+c_2+d_2 & b_2+c_2+d_2 \\ a_3+b_3+c_3 & a_3+b_3+d_3 & a_3+c_3+d_3 & b_3+c_3+d_3 \\ a_4+b_4+c_4 & a_4+b_4+d_4 & a_4+c_4+d_4 & b_4+c_4+d_4 \end{vmatrix}.$$

12. Generalize the preceding example. (See Ex. 24, after Art. 28.)

13. By means of the product,

$$\begin{vmatrix} a+b\sqrt{-1} & -c+d\sqrt{-1} \\ c+d\sqrt{-1} & a-b\sqrt{-1} \end{vmatrix} \cdot \begin{vmatrix} \alpha+\beta\sqrt{-1} & -\gamma+\delta\sqrt{-1} \\ \gamma+\delta\sqrt{-1} & \alpha-\beta\sqrt{-1} \end{vmatrix},$$

show that the product of two numbers, each the sum of four squares, is itself the sum of four squares.

14. By writing each factor as a determinant and performing the indicated multiplication, show that the product,

$$(\beta^2+\gamma^2)(\alpha^2+\gamma^2)(\alpha^2+\beta^2),$$

is the sum of two squares.

15. Express the product of two determinants of the second order in three distinct forms. (See Art. 55.)

Reciprocal Determinants.

59. If the elements of a determinant are replaced by their respective co-factors, the new determinant thus formed is called the **reciprocal** of the given determinant.

Thus, the reciprocal of

$$\delta \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the determinant

$$\Delta \equiv \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix}, \quad . \quad . \quad . \quad . \quad (2)$$

A_{ks} in Δ being the co-factor of the element a_{ks} in δ.

The reciprocal of a determinant is also called its *determinant adjugate.*

60. THEOREM.—*The reciprocal of any determinant of the* n*th order is equal to the* (n − 1)*th power of the given determinant.*

This may be shown as follows:

Multiplying together Equations (1) and (2) of the last article, member for member, we have

$$\delta\cdot\Delta=\begin{vmatrix} a_{11}A_{11}+\cdots+a_{1n}A_{1n} & a_{11}A_{21}+\cdots+a_{1n}A_{2n} & \cdots & a_{11}A_{n1}+\cdots+a_{1n}A_{nn} \\ a_{21}A_{11}+\cdots+a_{2n}A_{1n} & a_{21}A_{21}+\cdots+a_{2n}A_{2n} & \cdots & a_{21}A_{n1}+\cdots+a_{2n}A_{nn} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}A_{11}+\cdots+a_{nn}A_{1n} & a_{n1}A_{21}+\cdots+a_{nn}A_{2n} & \cdots & a_{n1}A_{n1}+\cdots+a_{nn}A_{nn} \end{vmatrix}.$$

By means of the formulæ in Articles 27 and 28, the above equation may be reduced to

$$\delta\cdot\Delta=\begin{vmatrix} \delta & 0 & \cdots & 0 \\ 0 & \delta & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \delta \end{vmatrix}=\delta^n, \text{ or}$$

$$\Delta=\delta^{n-1},$$

which is the theorem.

61. THEOREM.—*Any minor of the* m*th order in the reciprocal of a given determinant is equal to the product of the co-factor of the corresponding minor in the given determinant by the* (m − 1)*th power of the given determinant.*

Assuming

$$\delta\equiv|\, a_{11} \quad a_{22} \cdots a_{nn}\,|$$

as the given determinant, let the elements of the minor be taken from the fth, gth, hth, $\cdots$ rows and from the pth, qth, rth, $\cdots$ columns of the reciprocal determinant

$$\Delta \equiv | A_{11} \quad A_{22} \cdots A_{nn} |.$$

Also let

$$f+g+h+\cdots+p+q+r+\cdots=u.$$

The minor in question, which we assume to be of the mth order, may be written in the form (Art. 30)

$$\Delta_m = \begin{vmatrix} A_{fp} & A_{fq} & A_{fr} & \cdots & A_{f,m+1} & \cdots & A_{fn} \\ A_{gp} & A_{gq} & A_{gr} & \cdots & A_{g,m+1} & \cdots & A_{gn} \\ A_{hp} & A_{hq} & A_{hr} & \cdots & A_{h,m+1} & \cdots & A_{hn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 1 \end{vmatrix} . \quad . \quad . \quad (1)$$

In the given determinant let us pass the above numbered rows upwards and the above numbered columns to the left. In this manner we obtain (Art. 35)

$$\delta = (-1)^u \begin{vmatrix} a_{fp} & a_{fq} & a_{fr} & \cdots & a_{f,m+1} & \cdots & a_{fn} \\ a_{gp} & a_{gq} & a_{gr} & \cdots & a_{g,m+1} & \cdots & a_{gn} \\ a_{hp} & a_{hq} & a_{hr} & \cdots & a_{h,m+1} & \cdots & a_{hn} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m+1,p} & a_{m+1,q} & a_{m+1,r} & \cdots & a_{m+1,m+1} & \cdots & a_{m+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{np} & a_{nq} & a_{nr} & \cdots & a_{n,m+1} & \cdots & a_{nn} \end{vmatrix} . \quad . \quad (2)$$

Multiplying together Equations (1) and (2), member for member, gives

$$\Delta_m \cdot \delta = (-1)^u \begin{vmatrix} \delta & 0 & 0 & \cdots 0 & \cdots 0 \\ 0 & \delta & 0 & \cdots 0 & \cdots 0 \\ 0 & 0 & \delta & \cdots 0 & \cdots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{f,m+1} & a_{g,m+1} & a_{h,m+1} & \cdots a_{m+1,m+1} & \cdots a_{n,m+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{fn} & a_{gn} & a_{hn} & \cdots a_{m+1,n} & \cdots a_{nn} \end{vmatrix}$$

$$= (-1)^u \delta^m \begin{vmatrix} a_{m+1,m+1} & \cdots & a_{n,m+1} \\ \cdot & \cdot & \cdot \\ a_{m+1,n} & \cdots & a_{nn} \end{vmatrix}, \text{ or}$$

$$\Delta_m = \delta^{m-1}(-1)^u \begin{vmatrix} a_{m+1,m+1} & \cdots & a_{m+1,n} \\ \cdot & \cdot & \cdot \\ a_{n,m+1} & \cdots & a_{nn} \end{vmatrix}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which establishes the theorem, since the minor of δ in the second member is, with the sign-factor $(-1)^u$, the co-factor of the minor corresponding to Δ_m.

62. If $\delta = 0$, Equation (3) of the last article gives, for all values of m greater than unity, $\Delta_m = 0$. That is,

If a determinant is equal to zero, all the minors of its reciprocal which are of an order higher than the first are also equal to zero. (See Examples 19, 20, and 21, after Article 50.)

If $m=2$, we have for Δ_m any one of the following:

$$\begin{vmatrix} A_{fp} & A_{fq} \\ A_{gp} & A_{gq} \end{vmatrix} = \begin{vmatrix} A_{fq} & A_{fr} \\ A_{gq} & A_{gr} \end{vmatrix} = \cdots = 0.$$

Then,

$$A_{fp} : A_{gp} :: A_{fq} : A_{gq} :: A_{fr} : A_{gr} :: \cdots.$$

That is, *if a determinant is equal to zero, the co-factors of the elements of any row are proportional to the co-factors of the corresponding elements of any other row.* (See Art. 43.)

63. The theorem in Article 61 gives, employing the notation explained in Article 36,

$$\begin{vmatrix} A_{fp} & A_{fq} \\ A_{gp} & A_{gq} \end{vmatrix} = \delta \cdot \text{co-} \begin{vmatrix} a_{fp} & a_{fq} \\ a_{gp} & a_{gq} \end{vmatrix}, \quad . \quad . \quad . \quad (1)$$

or, in the notation of the calculus (Art. 39),

$$\begin{vmatrix} \dfrac{d\delta}{da_{fp}} & \dfrac{d\delta}{da_{fq}} \\ \dfrac{d\delta}{da_{gp}} & \dfrac{d\delta}{da_{gq}} \end{vmatrix} = \delta \cdot \frac{d^2\delta}{da_{fp}da_{gq}};$$

whence, by expansion,

$$\delta \frac{d^2\delta}{da_{fp}da_{gq}} = \frac{d\delta}{da_{fp}} \cdot \frac{d\delta}{da_{gq}} - \frac{d\delta}{da_{gp}} \cdot \frac{d\delta}{da_{fq}}. \quad . \quad . \quad (2)$$

CHAPTER VII.

DETERMINANTS OF SPECIAL FORMS.

There are certain classes of determinants which, by virtue of some mutual relation among the elements, or of some particular disposition of the same, possess properties peculiar to themselves. In the present chapter a few of the more important of these special forms of determinants will be examined.

Symmetrical Determinants.

64. Two elements of a determinant so situated that the row and column numbers of one are respectively the column and row numbers of the other are called **conjugate** elements; thus, in the determinant $| a_{11} \; a_{22} \cdots a_{nn} |$, the two elements a_{sk} and a_{ks} are conjugate to each other. Each element of the principal diagonal must be regarded as its own conjugate. In the same manner we may speak of conjugate co-factors of a determinant; A_{sk} and A_{ks} being such in reference to the determinant just written.

The term *conjugate* must not, in this connection, be confounded with the term *complementary*.

A row and a column having the same number are sometimes called conjugate lines.

65. If each element of a determinant is equal to its conjugate the determinant is said to be **axi-symmetric**; thus, the determinant

$$\Delta \equiv \begin{vmatrix} \alpha & a & b & c \\ a & \beta & l & m \\ b & l & \gamma & n \\ c & m & n & \delta \end{vmatrix}$$

is axi-symmetric.

The principal diagonal is called the **axis of symmetry**. If the elements of the axis are zeros the determinant is said to be **zero-axial**.

If, in $| a_{11}a_{22} \cdots a_{nn} |$,

$$a_{ks} = a_{n-s+1,\, n-k+1},$$

the determinant is symmetrical with respect to the secondary diagonal. Such a determinant is

$$\begin{vmatrix} z & y & x & 0 \\ q & p & 0 & x \\ r & 0 & p & y \\ 0 & r & q & z \end{vmatrix},$$

which is also zero-axial. Any determinant in this

form may be transformed to one which is symmetrical with respect to the principal diagonal by simply reversing the order of the columns (or rows), the sign being changed unless, n being the order of the determinant, either n or $(n-1)$ is divisible by *four*.

The following properties of axi-symmetric determinants may be deduced directly from the definition:

(a) *Conjugate lines are alike.*

(b) *Conjugate minors are equal.*

(c) *Minors which are co-axial with the given determinant are axi-symmetric.*

(d) *The reciprocal determinant is axi-symmetric.*

66. THEOREM.— *The square of any determinant is an axi-symmetric determinant.*

This follows directly from Formula (2) of Article 53. For if $a_{rl} = b_{rl}$ for all values of r and l from 1 to n, the formula referred to gives

$$\begin{aligned} p_{ks} &= a_{k1}a_{s1} + a_{k2}a_{s2} + \cdots + a_{kn}a_{sn} \\ &= a_{s1}a_{k1} + a_{s2}a_{k2} + \cdots + a_{sn}a_{kn} = p_{sk}, \end{aligned}$$

which establishes the theorem.

67. THEOREM.— *If an axi-symmetric determinant be multiplied by the square of any determinant of the same order the product will be an axi-symmetric determinant.*

Let $| a_{11} a_{22} \cdots a_{nn} |$ be the given axi-symmetric determinant, and $| b_{11} b_{22} \cdots b_{nn} |$ the determinant by the square of which it is to be multiplied. Also let

$$| b_{11} \; b_{22} \cdots b_{nn} | \cdot | a_{11} \; a_{22} \cdots a_{nn} | = | p_{11} \; p_{22} \cdots p_{nn} |.$$

We are to prove that, if

$$| P_{11} \; P_{22} \cdots P_{nn} | = | p_{11} \; p_{22} \cdots p_{nn} | \cdot | b_{11} \; b_{22} \cdots b_{nn} |,$$

then is $| P_{11} P_{22} \cdots P_{nn} |$ axi-symmetric; that is, $P_{ks} = P_{sk}$.

By Formula (2) of Article 53, we have

$$\begin{aligned} P_{ks} &= p_{k1} b_{s1} + p_{k2} b_{s2} + \cdots + p_{kn} b_{sn} \\ &= (b_{k1} a_{11} + b_{k2} a_{12} + \cdots + b_{kn} a_{1n}) b_{s1} \\ &\quad + (b_{k1} a_{21} + b_{k2} a_{22} + \cdots + b_{kn} a_{2n}) b_{s2} \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\quad + (b_{k1} a_{n1} + b_{k2} a_{n2} + \cdots + b_{kn} a_{nn}) b_{sn}, \end{aligned}$$

or, since $a_{rl} = a_{lr}$

$$\begin{aligned} P_{ks} &= (b_{s1} a_{11} + b_{s2} a_{21} + \cdots + b_{sn} a_{n1}) b_{k1} \\ &\quad + (b_{s1} a_{21} + b_{s2} a_{22} + \cdots + b_{sn} a_{2n}) b_{k2} \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\quad + (b_{s1} a_{n1} + b_{s2} a_{n2} + \cdots + b_{sn} a_{nn}) b_{kn} \\ &= p_{s1} b_{k1} + p_{s2} b_{k2} + \cdots + p_{sn} b_{kn} = P_{sk}, \end{aligned}$$

which establishes the theorem.

68. Corollary. — *All powers of an axi-symmetric determinant, and the even powers of any determinant, are axi-symmetric determinants.*

This readily follows from the theorems demonstrated in the last two articles.

EXAMPLE.

Tell whether or not the product of two axi-symmetric determinants of the same order is axi-symmetric.

69. If all the elements on every line perpendicular to a diagonal of a determinant are equal, the determinant is said to be **per-symmetric** or **ortho-symmetric**; thus, the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & a_2 & a_3 & \cdots a_n \\ a_2 & a_3 & a_4 & \cdots a_{n+1} \\ a_3 & a_4 & a_5 & \cdots a_{n+2} \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ a_n & a_{n+1} & a_{n+2} & \cdots a_{2n-1} \end{vmatrix} \quad . \quad . \quad . \quad . \quad (1)$$

is per-symmetric with respect to its principal diagonal.

The whole number of different elements in the above array being $(2n-1)$, we may write the per-symmetric determinant of the nth order in the abbreviated form

$$P(a_1, a_2, \cdots a_{2n-1}), \quad . \quad . \quad . \quad . \quad (2)$$

which is identical with (1).

70. Let the elements of the per-symmetric determinant given in the last article be written down *seriatim*, and the successive difference series formed; thus:

$$\begin{array}{ccccccccccc}
a_1 & & a_2 & & a_3 & & a_4 & & \cdots & & \cdots \\
& a_2 - a_1 & & a_3 - a_2 & & a_4 - a_3 & & \cdots & & \cdots & \\
& & a_3 - 2a_2 + a_1 & & a_4 - 2a_3 + a_2 & & \cdots & & \cdots & & \\
& & & a_4 - 3a_3 + 3a_2 - a_1 & & \cdots & & \cdots & & & \\
& & & & \cdots & & \cdots & & & & \\
& & & & & \cdots & & & & &
\end{array}$$

Introducing an obvious system of abbreviated notation, the above array may be written

$$\begin{array}{ccccccccccc}
a_1 & & a_2 & & a_3 & & a_4 & & \cdots & & \cdots \\
& \Delta^{(1)}a_1 & & \Delta^{(1)}a_2 & & \Delta^{(1)}a_3 & & \cdots & & \cdots & \\
& & \Delta^{(2)}a_1 & & \Delta^{(2)}a_2 & & \cdots & & \cdots & & \\
& & & \Delta^{(3)}a_1 & & \cdots & & \cdots & & & \\
& & & & \cdots & & \cdots & & & & \\
& & & & & \cdots & & & & &
\end{array}$$

We shall now demonstrate the following

71. THEOREM. — *The per-symmetric determinant* $P(a_1, a_2, \cdots a_{2n-1})$ *is equal to the per-symmetric determinant* $P(a_1, \Delta^{(1)}a_1, \Delta^{(2)}a_1, \cdots \Delta^{(2n-2)}a_1)$.

Replacing each element of the determinant Δ of Article 69 by that element minus the corresponding element in the preceding row; in the determinant

thus formed, replacing each element of each row after the second by that element minus the corresponding one in the preceding row; and continuing this process till finally the elements of the last row are the only ones operated upon, the resulting determinant may be written (Art. 22)

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots a_n \\ \Delta^{(1)}a_1 & \Delta^{(1)}a_2 & \Delta^{(1)}a_3 & \cdots \Delta^{(1)}a_n \\ \Delta^{(2)}a_1 & \Delta^{(2)}a_2 & \Delta^{(2)}a_3 & \cdots \Delta^{(2)}a_n \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ \Delta^{(n-1)}a_1 & \Delta^{(n-1)}a_2 & \Delta^{(n-1)}a_3 & \cdots \Delta^{(n-1)}a_n \end{vmatrix} . \quad . \quad . (1)$$

Operating in precisely the same manner upon the columns of the determinant (1), we have finally

$$\Delta = \begin{vmatrix} a_1 & \Delta^{(1)}a_1 & \Delta^{(2)}a_1 & \cdots \Delta^{(n-1)}a_1 \\ \Delta^{(1)}a_1 & \Delta^{(2)}a_1 & \Delta^{(3)}a_1 & \cdots \Delta^{(n)}a_1 \\ \Delta^{(2)}a_1 & \Delta^{(3)}a_1 & \Delta^{(4)}a_1 & \cdots \Delta^{(n+1)}a_1 \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ \Delta^{(n-1)}a_1 & \Delta^{(n)}a_1 & \Delta^{(n+1)}a_1 & \cdots \Delta^{(2n-2)}a_1 \end{vmatrix}, \quad . \quad . \quad (2)$$

which we were to prove.

It will be observed that the determinant (2) is also per-symmetric.

72. If each row of a determinant may be derived from the preceding row by passing the last element

over all the others to the first place, the determinant is called a **circulant**; thus,

$$\Delta \equiv \begin{vmatrix} a_1 & a_2 & a_3 \cdots a_n \\ a_n & a_1 & a_2 \cdots a_{n-1} \\ a_{n-1} & a_n & a_1 \cdots a_{n-2} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ a_2 & a_3 & a_4 \cdots a_1 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (1)$$

is a circulant.

The notation

$$\Delta \equiv C(a_1, a_2, \cdots a_n) \quad . \quad . \quad . \quad (2)$$

is very generally employed.

Circulants are obviously per-symmetric determinants, but they possess some properties not possessed by per-symmetric determinants in general. The most important of these is stated in the following

73. THEOREM.— *The determinant* $C(a_1, a_2, \cdots a_n)$ *contains as a factor* $a_1+a_2r+a_3r^2+\cdots+a_nr^{n-1}$, *in which* r *is one of the roots of the equation* $x_n-1=0$.

Let us consider the product

$$(a_1+a_2r+a_3r^2+\cdots+a_nr^{n-1})(A_1+A_2r^{-1}+A_3r^{-2}+\cdots+A_nr^{-n+1}).$$

For the coefficient of r^{k-1} in the above product we have, remembering that $r^{k-i}\equiv r^{n+k-i}$,

$$a_kA_1+a_{k+1}A_2+\cdots+a_{k-1}A_n.$$

When $k=1$, this coefficient becomes

$$a_1A_1+a_2A_2+\cdots+a_nA_n,$$

which is equal to the given determinant; but for all other values of k it vanishes (Arts. 27 and 28).

Hence the given determinant contains the factor

$$a_1+a_2r+a_3r^2+\cdots+a_nr^{n-1}.$$

74. If we let

$$\phi(r)=a_1+a_2r+\cdots+a_nr^{n-1},$$

and represent the different roots of $x^n-1=0$ by $r_1, r_2, \cdots r_n$, we may, by the preceding article, write

$$C(a_1, a_2, \cdots a_n)=\phi(r_1)\cdot\phi(r_2)\cdots\phi(r_n).$$

Since one of the values of r is unity,

$$C(a_1, a_2, \cdots a_n)$$

is divisible by $a_1+a_2+\cdots+a_n$, that is:

A circulant is divisible by the sum of its elements.

Let the student show this by another method.

Skew Determinants.

75. If each element on one side of the diagonal of a determinant is equal to its conjugate with its sign changed, the determinant is called a **skew determinant**; thus,

$$\Delta \equiv \begin{vmatrix} \alpha & a & b & c \\ -a & \beta & l & m \\ -b & -l & \gamma & n \\ -c & -m & -n & \delta \end{vmatrix} \quad \ldots\ldots (1)$$

is a skew determinant.

If every element of a determinant is equal to its conjugate with its sign changed, the determinant is defined as **skew-symmetric.** Since the elements on the principal diagonal are their own conjugates, it follows that skew-symmetric determinants are zero-axial. The determinant

$$\Delta \equiv \begin{vmatrix} 0 & a & b & c \\ -a & 0 & l & m \\ -b & -l & 0 & n \\ -c & -m & -n & 0 \end{vmatrix} \quad \ldots\ldots (2)$$

is skew-symmetric.

76. The following properties of skew-symmetric determinants correspond to those of axi-symmetric determinants given in Article 65:

(a) *Conjugate lines differ only in the signs of their elements.*

(b) *Conjugate minors are equal, or differ only in sign.*

(c) *Minors which are co-axial with the given determinant are themselves skew-symmetric.*

(d) *The reciprocal determinant is skew when of even order, but axi-symmetric when of odd order.*

This last property may be proved as follows:

Let the given determinant be $| a_{11}\ a_{22} \cdots a_{nn} |$, and its reciprocal $| A_{11}\ A_{22} \cdots A_{nn} |$.

The co-factor A_{ks} differs from the co-factor A_{sk} in the sign of every element; hence,

$$A_{ks} = (-1)^{n-1} A_{sk}.$$

If n is even, $A_{ks} = -A_{sk}$, but if n is odd, $A_{ks} = A_{sk}$; or, the reciprocal determinant is skew when n is even, axi-symmetric when n is odd.

77. THEOREM. — *A skew-symmetric determinant of odd order is equal to zero.*

For, multiplying each element by -1 simply changes columns to rows and rows to columns, which does not change the value of the determinant. Hence, letting δ be the given determinant and n its order,

$$\delta = (-1)^n \delta,$$

which, if n is odd, can only be true when $\delta = 0$.

EXAMPLE.

Show that the reciprocal of a skew-symmetric determinant of even order is not only skew, but skew-symmetric.

78. THEOREM. — *A skew-symmetric determinant of even order is the square of some rational function of its elements.*

Let us represent the given skew-symmetric determinant, which we assume to be of the $2n$th order, by Δ. Also, let a_{rl} be any element of the given determinant and A_{rl} the corresponding element of its reciprocal.

Then, by Equation (1) of Article 63, we have

$$\begin{vmatrix} A_{rr} & A_{rl} \\ A_{lr} & A_{ll} \end{vmatrix} = \Delta \cdot \text{co-}\begin{vmatrix} a_{rr} & a_{rl} \\ a_{lr} & a_{ll} \end{vmatrix}. \quad . \quad . \quad . \quad (1)$$

But, since Δ is skew-symmetric,

$$\left.\begin{aligned} a_{rl} &= -a_{lr}, \\ a_{rr} &= 0 = a_{ll}, \\ A_{rl} &= -A_{lr}, \\ A_{rr} &= 0 = A_{ll}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By means of (2) we may write Equation (1) in the form

$$\Delta = \frac{(A_{rl})^2}{\text{co-}\begin{vmatrix} 0 & a_{rl} \\ -a_{rl} & 0 \end{vmatrix}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The denominator in the second member of the above equation is, by Article 76 (c), a skew-symmetric determinant of the order $(2n-2)$, and, by Article 35, its sign factor is $(-1)^{2(r+l)}$ or $+1$.

Since Δ is, by Equation (3), a perfect square when the denominator of the second member is such, it follows that if any skew-symmetric determinant of even order is a perfect square, that of the second higher order must also be a perfect square. Now, any skew-symmetric determinant of the second order is of the form

$$\begin{vmatrix} 0 & e \\ -e & 0 \end{vmatrix},$$

which is a perfect square. Hence, by induction, any skew-symmetric determinant of even order is a perfect square; which is the theorem.

As an illustration, let us resume the determinant (2) of Article 75. By Equation (3), making $r=4$ and $l=1$, this determinant may be written

$$\Delta = \frac{\left\{-\begin{vmatrix} a & b & c \\ 0 & l & m \\ -l & 0 & n \end{vmatrix}\right\}^2}{\begin{vmatrix} 0 & l \\ -l & 0 \end{vmatrix}}, \text{ or}$$

$$\Delta = (an - bm + cl)^2.$$

Pfaffians.

79. The last article introduces us to a class of functions known as Pfaffians, which we now proceed to examine.

In the skew-symmetric determinant of even order

$$\Delta \equiv \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & \cdots & a_{1,\,2n-1} & a_{1,\,2n} \\ -a_{12} & 0 & a_{23} & a_{24} & \cdots & a_{2,\,2n-1} & a_{2,\,2n} \\ -a_{13} & -a_{23} & 0 & a_{34} & \cdots & a_{3,\,2n-1} & a_{3,\,2n} \\ -a_{14} & -a_{24} & -a_{34} & 0 & \cdots & a_{4,\,2n-1} & a_{4,\,2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{1,\,2n-1} & -a_{2,\,2n-1} & -a_{3,\,2n-1} & -a_{4,\,2n-1} & \cdots & 0 & a_{2n-1,\,2n} \\ -a_{1,\,2n} & -a_{2,\,2n} & -a_{3,\,2n} & -a_{4,\,2n} & \cdots & -a_{2n-1,\,2n} & 0 \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

one of the terms is

$$a^2_{12}a^2_{34}a^2_{56} \cdots a^2_{2n-1,\,2n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since the determinant Δ is a perfect square, one of its square roots contains the term

$$+a_{12}a_{34}a_{56} \cdots a_{2n-1,\,2n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

while the other contains the term

$$-a_{12}a_{34}a_{56} \cdots a_{2n-1,\,2n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

That square root of Δ which contains the positive term (3) is called the **Pfaffian*** of the elements above the zero axis. Thus, in the preceding article,

$$an - bm + cl$$

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* These functions were named by Professor Cayley after the mathematician Jean-Frédéric Pfaff (1765–1825), who proposed a problem of historic interest in the discussion of which they were used by Jacobi.

is the Pfaffian of the elements

$$\begin{array}{ccc} a, & b, & c, \\ & l, & m, \\ & & n. \end{array}$$

80. The Pfaffian related to the determinant (1) in the preceding article may be represented by the array

$$P \equiv \left| \begin{array}{cccccc} a_{12} & a_{13} & a_{14} & \cdots & a_{1,\,2n-1} & a_{1,\,2n} \\ & a_{23} & a_{24} & \cdots & a_{2,\,2n-1} & a_{2,\,2n} \\ & & a_{34} & \cdots & a_{3,\,2n-1} & a_{3,\,2n} \\ & & & \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \\ & & & & a_{2n-2,\,2n-1} & a_{2n-2,\,2n} \\ & & & & & a_{2n-1,\,2n} \end{array} \right| \quad . \quad . \quad . \quad (1)$$

However, the umbral notation

$$P \equiv {}^{1}\!| \; a_{12} \quad a_{23} \quad a_{34} \cdots a_{2n-1,\,2n} \; |, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or $$P \equiv ff(a_{12} \quad a_{23} \quad a_{34} \cdots a_{2n-1,\,2n}) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is more convenient. These are still further abbreviated to

$$P \equiv {}^{1}\!| \; a_{1,\,2n} \; | \text{ and } P = ff(a_{1,\,2n}), \quad . \quad . \quad . \quad (4), (5)$$

respectively. The term

$$a_{12}a_{34}a_{56} \cdots a_{2n-1,\,2n}$$

is called the principal term of the above Pfaffian.

81. The order of a Pfaffian is the degree of its terms in reference to its elements. Since a Pfaffian is the square root of a skew-symmetric determinant, its order is one half that of the determinant to which it is thus related.* Thus the Pfaffian P of the preceding article is of the nth order.

The definitions of the terms *column*, *row*, and *element* as applied to determinants answer equally well for Pfaffians. As in determinants, any element may be defined by its row and column numbers, but this method is usually modified as follows:

The first row of a Pfaffian of the nth order contains $(2n-1)$ elements and is called the first **frame line.** The line through the first column and the second row also contains $(2n-1)$ elements and is called the second frame line. Similarly, the lines through the second column and the third row, the third column and the fourth row, ..., and through the last column, contain $(2n-1)$ elements each, and are called respectively the third, fourth, ..., and $2n$th frame lines.

The position of any element is defined by giving the numbers of the two frame lines intersecting in the element in question.

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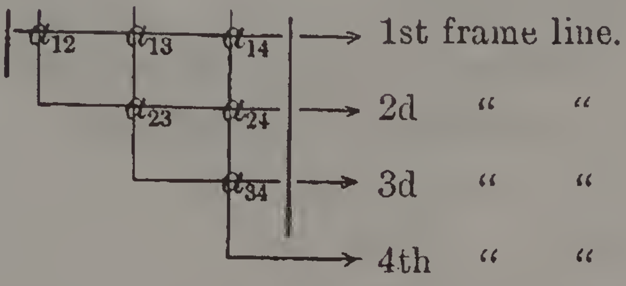
* Some writers regard the order of a Pfaffian as being the same as that of the corresponding skew-symmetric determinant.

This system is illustrated by the following array:



It should be noticed that the element in the lth and rth frame lines of a Pfaffian is the same as the element in the lth row and the rth column ($l<r$) of the skew-symmetric determinant to which the Pfaffian is related.

82. If the two frame lines of any element of a Pfaffian array be deleted and the remaining elements left undisturbed, the result is called the *minor* of the given element. The minor of the element a_{rl} of the Pfaffian P may be written $P_{(r,l}$.

If Δ represents the determinant (1) of Article 79, and P the Pfaffian (1) of Article 80, it may be seen by inspection that $P_{(r,l}$ is the Pfaffian related to the determinant $\Delta^{(r,l}_{(r,l}$. If Δ represents a skew-symmetric determinant of *odd* order, then is $\Delta^{(r}_{(r}$ of *even* order. The Pfaffian related to $\Delta^{(r}_{(r}$ may be represented by $P_{(r}$.

83. THEOREM. — *The determinant formed by bordering a skew-symmetric determinant with a new*

column and a new row may be expressed as the product of two Pfaffians.

(a) We shall first consider the case in which the bordered determinant is of odd order.

In the determinant Δ of Article 79, which is of even order $2n$, delete the first column and the last row. This gives the bordered skew-symmetric determinant of odd order,

$$\Delta_{\{2n}^{\{1} \equiv \begin{vmatrix} a_{12} & a_{13} & a_{14} & \cdots & a_{1,\,2n} \\ 0 & a_{23} & a_{24} & \cdots & a_{2,\,2n} \\ -a_{23} & 0 & a_{34} & \cdots & a_{3,\,2n} \\ -a_{24} & -a_{34} & 0 & \cdots & a_{4,\,2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{2,\,2n-1} & -a_{3,\,2n-1} & -a_{4,\,2n-1} & \cdots & a_{2n-1,\,2n} \end{vmatrix} \quad . \; . \; (1)$$

Deleting the first row and the last column of the determinant just written, leaves us

$$\Delta_{\{2n,\,1}^{\{1,\,2n} \equiv \begin{vmatrix} 0 & a_{23} & a_{24} & \cdots & a_{2,\,2n-1} \\ -a_{23} & 0 & a_{34} & \cdots & a_{3,\,2n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{2,\,2n-2} & -a_{3,\,2n-2} & -a_{4,\,2n-2} & \cdots & a_{2n-2,\,2n-1} \\ -a_{2,\,2n-1} & -a_{3,\,2n-1} & -a_{4,\,2n-1} & \cdots & 0 \end{vmatrix}, \quad (2)$$

which is a skew-symmetric determinant of even order.

By Equation (3) of Article 78 we have

$$\{\Delta_{\{2n}^{\{1}\}^2 = \Delta \cdot \Delta_{\{2n,\,1}^{\{1,\,2n}. \quad . \; . \; . \; . \; . \; . \; (3)$$

Taking the square root of both members of this equation gives

$$\Delta_{2n}^{1} = \begin{vmatrix} a_{12} & a_{13} & a_{14} & \cdots & a_{1,2n} \\ & a_{23} & a_{24} & \cdots & a_{2,2n} \\ & & a_{34} & \cdots & a_{3,2n} \\ & & \cdot & \cdot \quad \cdot & \cdot \\ & & & & a_{2n-1,2n} \end{vmatrix} \cdot \begin{vmatrix} a_{23} & a_{24} & \cdots & a_{2,2n-1} \\ & a_{34} & \cdots & a_{3,2n-1} \\ & \cdot & \cdot \quad \cdot & \cdot \\ & & & a_{2n-2,2n-1} \end{vmatrix}, \quad . . (4)$$

which establishes the theorem for the case in which the given determinant is of odd order.

(*b*) In case the given bordered determinant is of even order, let us consider it as derived from the skew-symmetric determinant

$$\Delta' \equiv \begin{vmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1,2n-1} & 0 \\ -a_{12} & 0 & a_{23} & \cdots & a_{2,2n-1} & 0 \\ -a_{13} & -a_{23} & 0 & \cdots & a_{3,2n-1} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_{1,2n-1} & -a_{2,2n-1} & -a_{3,2n-1} & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{vmatrix}, \quad . . (5)$$

by deleting the first column and the last row, and then reducing by Article 29. This gives

$$\Delta'^{1}_{2n} = \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,2n-1} \\ 0 & a_{23} & \cdots & a_{2,2n-1} \\ -a_{23} & 0 & \cdots & a_{3,2n-1} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{2,2n-2} & -a_{3,2n-2} & \cdots & a_{2n-2,2n-1} \end{vmatrix}, \quad . \quad . \quad . (6)$$

which is a bordered skew-symmetric determinant of even order.

Since Δ' is skew-symmetric, we have, by Equation (3) of Article 78,

$$\left\{\Delta'\begin{Bmatrix}1\\2n\end{Bmatrix}\right\}^2 = \Delta' \cdot \Delta'\begin{Bmatrix}1, 2n\\2n, 1\end{Bmatrix}. \qquad (7)$$

Now let Δ represent the skew-symmetric determinant $\Delta'\begin{Bmatrix}2n\\2n\end{Bmatrix}$, which is of odd order $(2n-1)$. Then

$$\left.\begin{aligned}&\Delta'\begin{Bmatrix}1\\2n\end{Bmatrix} \equiv \Delta\begin{Bmatrix}1\\2n-1\end{Bmatrix}, \quad \Delta'\begin{Bmatrix}1, 2n\\2n, 1\end{Bmatrix} \equiv \Delta\begin{Bmatrix}1\\1\end{Bmatrix},\\&\Delta' = \Delta'\begin{Bmatrix}2n, 2n-1\\2n, 2n-1\end{Bmatrix} \equiv \Delta\begin{Bmatrix}2n-1\\2n-1\end{Bmatrix}. \quad \text{(See Art. 29.)}\end{aligned}\right\} \ldots (8)$$

Substituting these values in Equation (7) gives

$$\left\{\Delta\begin{Bmatrix}1\\2n-1\end{Bmatrix}\right\}^2 = \Delta\begin{Bmatrix}2n-1\\2n-1\end{Bmatrix} \cdot \Delta\begin{Bmatrix}1\\1\end{Bmatrix}, \qquad (9)$$

in which the two determinants in the second member are skew-symmetric and of even order. Taking the square root of both members, we have

$$\Delta\begin{Bmatrix}1\\2n-1\end{Bmatrix} = \begin{vmatrix} a_{12} & a_{13} \cdots & a_{1,\,2n-2} \\ & a_{23} \cdots & a_{2,\,2n-2} \\ & \cdot \quad \cdot & \cdot \quad \cdot \\ & & a_{2n-3,\,2n-2} \end{vmatrix} \cdot \begin{vmatrix} a_{23} & a_{24} \cdots & a_{2,\,2n-1} \\ & a_{34} \cdots & a_{3,\,2n-1} \\ & \cdot \quad \cdot & \cdot \quad \cdot \\ & & a_{2n-2,\,2n-1} \end{vmatrix}, \quad (10)$$

which completes the demonstration of the theorem, since $\Delta\begin{Bmatrix}1\\2n-1\end{Bmatrix}$ is a bordered skew-symmetric determinant and is of even order.

84. Equation (4) of the last article may be written

$$\Delta_{(2n}^{(1} = P \cdot P_{(2n,\,1)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which Δ is a skew-symmetric determinant of even order $2n$, and P its Pfaffian. Also, Equation (10) of the same article may be written

$$\Delta_{(2n-1}^{(1} = P_{(2n-1}\, P_{(1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which Δ is a skew-symmetric determinant of odd order, $P_{(2n-1}$ the Pfaffian of $\Delta_{(2n-1}^{(2n-1}$, and $P_{(1}$ the Pfaffian of $\Delta_{(1}^{(1}$.

Since Equations (1) and (2) depend upon the general Equation (3) of Article 78, they may themselves be generalized, giving respectively

$$\Delta_{(r}^{(t} = P \cdot P_{(r,\,t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

when Δ is of even order, and

$$\Delta_{(r}^{(t} = P_{(r} \cdot P_{(t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

when Δ is of odd order.

Equations (3) and (4) show that

The minor of any element outside the principal diagonal of a skew-symmetric determinant may be expressed as the product of two Pfaffians.

If Δ is a skew-symmetric determinant of even order, then is $\Delta_{(i}^{(i}$ skew-symmetric and of odd order. Then, by Equation (4)

$$\Delta_{(i,\,r}^{(i,\,l} = P_{(i,\,r} \cdot P_{(i,\,l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

85. THEOREM. — *A Pfaffian of the* n*th order is equal to the sum of the* (2n−1) *products formed by multiplying each element of the first frame line by its minor, these products to be taken alternately plus and minus, the first being plus.*

Expanding the determinant Δ of Article 79 in terms of the elements of the first column and the first row by Cauchy's method (Art. 38) gives

$$\begin{aligned}\Delta = {} & a_{12}^2 \Delta_{(1,\,2}^{(1,\,2} - 2\,a_{12}a_{13}\,\Delta_{(1,\,3}^{(1,\,2} + 2\,a_{12}a_{14}\,\Delta_{(1,\,4}^{(1,\,2} - \cdots + 2\,a_{12}a_{1,\,2n}\Delta_{(1,\,2n}^{(1,\,2} \\ & + a_{13}^2 \Delta_{(1,\,3}^{(1,\,3} - 2\,a_{13}a_{14}\,\Delta_{(1,\,4}^{(1,\,3} + \cdots - 2\,a_{13}a_{1,\,2n}\,\Delta_{(1,\,2n}^{(1,\,3} \\ & + \cdots \\ & + a_{1,\,2n}^2 \Delta_{(1,\,2n}^{(1,\,2n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)\end{aligned}$$

Representing the Pfaffian corresponding to Δ by P, we have

$$\Delta \quad = P^2, \quad \Delta_{(1,\,r}^{(1,\,r} = \{P_{1,\,r}\}^2,$$

and, by the preceding article,

$$\Delta_{(1,\,r}^{(1,\,l} = P_{(1,\,r} \cdot P_{(1,\,l}.$$

By means of these relations Equation (1) becomes

$$\begin{aligned}P^2 = {} & a_{12}^2\,\{P_{(1,\,2}\}^2 - 2\,a_{12}a_{13}P_{(1,\,2}P_{(1,\,3} + 2\,a_{12}a_{14}P_{(1,\,2}P_{(1,\,4} - \cdots \\ & + 2\,a_{12}a_{1,\,2n}P_{(1,\,2}P_{1,\,2n}\end{aligned}$$

$$+ a_{13}^2\{P_{(1,3)}\}^2 - 2a_{13}a_{14}P_{(1,3)}P_{(1,4)} + \cdots - 2a_{13}a_{1,2n}P_{(1,3)}P_{(1,2n)}$$
$$+ \cdots$$
$$+ a_{1,2n}^2\{P_{(1,2n)}\}^2. \quad . \quad . \quad . \quad (2)$$

Taking the square root of each member of this equation, we have

$$P = a_{12}P_{(1,2)} - a_{13}P_{(1,3)} + a_{14}P_{(1,4)} - \cdots + a_{1,2n}P_{(1,2n)} \quad . \quad . \quad (3)$$

which is the symbolic form of the theorem to be demonstrated.

Each of the minors $P_{(1,2)}$, $P_{(1,3)} \cdots P_{(1,2n)}$ may be expanded in the same manner as the original Pfaffian. We thus have a method of forming the complete expansion of a Pfaffian in terms of its elements analogous to the method of expanding determinants explained in Article 27.

The method is illustrated in the development of the following Pfaffian:

$$P = \left| \begin{matrix} a & b & c & d & e \\ & f & g & h & i \\ & & j & k & l \\ & & & m & n \\ & & & & o \end{matrix} \right|$$

$$= a \left| \begin{matrix} j & k & l \\ & m & n \\ & & o \end{matrix} \right| - b \left| \begin{matrix} g & h & i \\ & m & n \\ & & o \end{matrix} \right| + c \left| \begin{matrix} f & h & i \\ & k & l \\ & & o \end{matrix} \right|$$

$$-d\left|\begin{array}{ccc} f & g & i\\ & j & l\\ & & n\end{array}\right| + e\left|\begin{array}{ccc} f & g & h\\ & j & k\\ & & m\end{array}\right|$$

$$= a(jo - kn + lm) - b(go - hn + im) + c(fo - hl + ik)$$
$$- d(fn - gl + ij) + e(fm - gk + hj).$$

86. An inspection of the method of expansion explained in the preceding article shows that the number of terms in the complete expansion of a Pfaffian of the nth order is the product

$$1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1),$$

an odd number. Of these terms there is one more positive than negative.

EXAMPLES.

1. Expand $\left|\begin{array}{cccc} a & b & c & d\\ & e & f & g\\ & & h & i\\ & & & j\end{array}\right|$,

and explain the significance of the result.

2. Prove that if any two frame lines of a Pfaffian be interchanged and the element at their intersection changed in sign, the value of the Pfaffian is changed only in sign.

State and prove a property of Pfaffians analogous to that established for determinants in

3. Article 20.

4. Article 60.

5. Article 61.

6. Article 62.

7. Show that a Pfaffian, one of whose frame lines contains only binomial elements, may be expressed as the sum of two Pfaffians.

8. By squaring both members of the result in the last example prove the theorem of Article 83.

9. Prove that

$$\Delta^{(l}_{(r} : \Delta^{(k}_{(s} :: P_{(r,\,l)} : P_{(s,\,k)}$$

Δ being a skew-symmetric determinant of even order of which the corresponding Pfaffian is P.

10. Prove that if an axi-symmetric determinant vanishes, the co-factor of any element is a mean proportional between the co-factors of the principal diagonal elements belonging to the row and column of the given element.

11. Prove that, if the minor of the leading element of an axi-symmetric determinant is zero, the determinant is expressible as a second power.

12. Prove that the co-factor of the sum of the elements of a circulant of the order $(2n-1)$ is expressible as a determinant of the nth order. (See Art. 74.)

Alternants.

87. A function of two or more variables such that the interchange of any two of the variables

changes the sign without changing the value of the function is called an *alternating function* of these variables.

Alternating functions are to be distinguished from *symmetric functions*, in which the interchange of two variables changes neither the sign nor the value of the function. As examples of alternating functions we give:

$$(x-y)^3,$$
$$x^2y - xy^2 + xz^2 - x^2z + y^2z - yz^2,$$
$$\sin(x^2 - y^2),$$
$$x \sin y - y \sin x,$$

and $$\log \frac{x}{y}.$$

On the other hand,

$$(x-y)^2,$$
$$x^2 + y^2 + z^2 + yz + xz + xy,$$
$$\cos(x^2 - y^2),$$
$$x \sin y + y \sin x,$$

and $$\log xy$$

are symmetrical.

88. The determinant

$$\begin{vmatrix} x_1 & e^{x_1} & \sin x_1 \\ x_2 & e^{x_2} & \sin x_2 \\ x_3 & e^{x_3} & \sin x_3 \end{vmatrix}$$

is an alternating function of x_1, x_2, and x_3, for the reason that interchanging any two of these variables amounts to the same thing as interchanging two rows of the above determinant, a process which only reverses the sign. Such a determinant as the above is called an alternant.

An **alternant** may be defined as a determinant in which the elements of the first row are functions of some variable, the corresponding elements of the second row the same functions of another variable, etc.

Any alternant whose elements are functions each of a single variable may be expressed as follows:

$$\Delta \equiv \begin{vmatrix} f_0(x_1) & f_1(x_1) & \cdots f_{n-1}(x_1) \\ f_0(x_2) & f_1(x_2) & \cdots f_{n-1}(x_2) \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ f_0(x_n) & f_1(x_n) & \cdots f_{n-1}(x_n) \end{vmatrix}, \quad . \quad . \quad . \quad (1)$$

for which has been adopted the notation

$$\Delta \equiv A\,[f_0(x_1),\quad f_1(x_2),\ \cdots f_{n-1}(x_n)].$$

A *simple alternant* is one in which the functions used as elements are powers of the variables.

89. Let us compare the simple alternant

$$A\,(x_1^0, x_2, x_3^2, \cdots\ x_n^{n-1}) \equiv \begin{vmatrix} 1 & x_1 & x_1^2 \cdots x_1^{n-1} \\ 1 & x_2 & x_2^2 \cdots x_2^{n-1} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ 1 & x_n & x_n^2 \cdots x_n^{n-1} \end{vmatrix},$$

with the product

$$\begin{aligned} P \equiv (x_2 - x_1)(x_3 - x_1)(x_4 - x_1) & \cdots (x_n - x_1) \\ \times (x_3 - x_2)(x_4 - x_2) & \cdots (x_n - x_2) \\ \times (x_4 - x_3) & \cdots (x_n - x_3) \\ & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & \times (x_n - x_{n-1}) ; \end{aligned}$$

that is, the product of the differences obtained by subtracting each of the n quantities $x_1, x_2, \cdots x_n$ from each of those which follow it.

The determinant vanishes upon any two of the quantities, as x_i and x_k, becoming equal, and is therefore divisible by $(x_k - x_i)$. Being thus divisible by any of the factors of P, the determinant is divisible by P. Since the determinant and P are of the same degree, $\frac{n}{2}(n-1)$, with respect to the quantities $x_1, x_2, \cdots x_n$, their quotient must be independent of these quantities. To obtain this quotient we may compare any term of the expanded product with that term of the expansion of the determinant in which $x_1, x_2, \cdots x_n$ are affected by the same exponents. The product of the first terms of all the binomial factors of P is

$$x_2 x_3^2 x_4^3 \cdots x_n^{n-1};$$

but this is also the leading term of the determinant. Hence, the quotient sought is unity, and

$$A(x_1^0, x_2, x_3^2, \cdots x_n^{n-1}) = P.$$

90. The product P is called the *difference product* of $x_1, x_2, \cdots x_n$, and for it the notation

$$P \equiv \zeta^{\frac{1}{2}}(x_1, \quad x_2, \cdots x_n)$$

has been adopted; in accordance with which

$$P^2 \equiv \zeta(x_1, \quad x_2, \cdots x_n),$$

the notation for the square of the difference product.

The alternant $A\ (x_1^0, x_2, x_3^2, \cdots x_n^{n-1})$ is called the difference product alternant.

91. In the alternant $A\,[f_0(x_1), f_1(x_2), \cdots f_{n-1}(x_n)]$ let $f_i(x)$ be, in each case, a rational integral function of the degree i; and a_i the coefficient of x^i, the highest power of x in $f_i(x)$.

Multiplying the elements of the first column, each of which must be represented by a_0, by the proper factor and subtracting from the corresponding elements of the second column, we have a new second column containing the elements a_1x_1, a_1x_2, $\cdots a_1x_n$. Then, subtracting the elements of the first and second columns, each multiplied by the proper factor, from the corresponding elements of the third column, we have a new third column made up of the elements $a_2x_1^2$, $a_2x_2^2$, $\cdots a_2x_n^2$. Continuing this process, we finally reduce the

given alternant, without change of value (Art. 22), to the form

$$A[f_0(x_1),\ f_1(x_2),\ \cdots f_{n-1}(x_n)]$$

$$=\begin{vmatrix} a_0 & a_1x_1 & a_2x_1^2 \cdots a_{n-1}x_1^{n-1} \\ a_0 & a_1x_2 & a_2x_2^2 \cdots a_{n-1}x_2^{n-1} \\ \cdot & \cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_0 & a_1x_n & a_2x_n^2 \cdots a_{n-1}x_n^{n-1} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, by the last article,

$$\frac{A[f_0(x_1), f_1(x_2), \cdots f_{n-1}(x_n)]}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)} = a_0a_1a_2 \cdots a_{n-1} \quad . \quad . \quad . (2)$$

This result is a special case of the theorem in the next article.

92. THEOREM. — *Every alternant whose elements are functions of* $x_1, x_2, \cdots x_n$ *is exactly divisible by* $\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)$, *and the quotient is a symmetric function of these quantities.*

For, the alternant vanishes upon any two of the quantities $x_1, x_2, \cdots x_n$, as x_i and x_k, becoming equal, and therefore contains the factor $x_k - x_i$, and consequently the product of all such factors, which is $\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)$. Moreover, since the interchange of x_i and x_k changes the sign, both of the alternant and of $\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)$, the quotient remains unaffected by any such interchange and is therefore a symmetric function.

93. We may express the coefficients of the rational integral function of x as symmetrical functions of the roots in the following manner:

If $r_1, r_2, \cdots r_n$ are the roots of such a function of x, we have

$$\begin{aligned} &x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n \\ &\quad = (x - r_1)(x - r_2)\cdots(x - r_n) = 0. \quad . \quad . \quad . \quad (1) \end{aligned}$$

Multiplying both members of the above equation by $\zeta^{\frac{1}{2}}(r_1, r_2, \cdots r_n)$, the result may be written

$$\begin{aligned} &\zeta^{\frac{1}{2}}(r_1, r_2, \cdots r_n)(x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n) \\ &= \xi^{\frac{1}{2}}(r_1, r_2, \cdots r_n, x) \\ &= \begin{vmatrix} 1 & r_1 & r_1^2 & \cdots & r_1^n \\ 1 & r_2 & r_2^2 & \cdots & r_2^n \\ . & . & . & . & . \\ 1 & r_n & r_n^2 & \cdots & r_n^n \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

Developing the determinant according to the elements of the last row and equating the coefficients of the same powers of x in the first and third members gives:

$$a_n = \frac{(-1)^n}{\zeta^{\frac{1}{2}}(r_1, r_2, \cdots r_n)} \begin{vmatrix} r_1 & r_1^2 & \cdots & r_1^n \\ r_2 & r_2^2 & \cdots & r_2^n \\ . & . & . & . \\ r_n & r_n^2 & \cdots & r_n^n \end{vmatrix};$$

$$a_{n-1} = \frac{(-1)^{n-1}}{\zeta^{\frac{1}{2}}(r_1, r_2, \cdots r_n)} \begin{vmatrix} 1 & r_1^2 & \cdots & r_1^n \\ 1 & r_2^2 & \cdots & r_2^n \\ . & . & . & . \\ 1 & r_n^2 & \cdots & r_n^n \end{vmatrix};$$

. . . .

$$a_{n-k} = \frac{(-1)^{n-k}}{\zeta^{\frac{1}{2}}(r_1, r_2, \cdots r_n)} \begin{vmatrix} 1 & r_1 & \cdots & r_1^{k-1} & r_1^{k+1} & \cdots & r_1^n \\ 1 & r_2 & \cdots & r_2^{k-1} & r_2^{k+1} & \cdots & r_2^n \\ . & . & . & . & . & . & . \\ 1 & r_n & \cdots & r_n^{k-1} & r_n^{k+1} & \cdots & r_n^n \end{vmatrix};$$

. . . .

$$a_1 = \frac{(-1)}{\zeta^{\frac{1}{2}}(r_1, r_2, \cdots r_n)} \begin{vmatrix} 1 & r_1 & \cdots & r_1^{n-2} & r_1^n \\ 1 & r_2 & \cdots & r_2^{n-2} & r_2^n \\ . & . & . & . & . \\ 1 & r_n & \cdots & r_n^{n-2} & r_n^n \end{vmatrix}.$$

The above functions of the roots are symmetric by Article 92.

94. Resuming Equation (1) of the preceding article, we have, by the theory of equations:

$$a_n \ = (-1)^n r_1 r_2 \cdots r_n;$$

$$a_{n-1} = (-1)^{n-1} \sum_{n-1} (+ r_p r_q \cdots r_t);$$

and, in general,

$$a_{n-k} = (-1)^{n-k} \sum_{n-k} (+ r_p r_q \cdots r_t),$$

in which $\sum\limits_{n-k}$ implies that the sum is to be taken of all terms of degree $(n-k)$ that can be formed from

$r_1, r_2, \cdots r_n$, no root to appear as a factor more than once in the same term.

Comparing the above value of a_{n-k} with that given in the preceding article, we readily obtain the general relation

$$\frac{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{k-1} & x_1^{k+1} & x_1^{k+2} & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^{k-1} & x_2^{k+1} & x_2^{k+2} & \cdots & x_2^n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & \cdots & x_n^{k-1} & x_n^{k+1} & x_n^{k+2} & \cdots & x_n^n \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{k-1} & x_1^{k} & x_1^{k+1} & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{k-1} & x_2^{k} & x_2^{k+1} & \cdots & x_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & \cdots & x_n^{k-1} & x_n^{k} & x_n^{k+1} & \cdots & x_n^{n-1} \end{vmatrix}} = \sum_{n-k} (+\, x_p x_q \cdots x_t).$$

95. Squaring the difference product $\zeta^{\frac{1}{2}}(a_1, a_2, \cdots a_n)$ by Formula (1) of Article 53, we have, representing $a_1^i + a_2^i + \cdots + a_n^i$ by s_i,

$$\zeta(a_1, a_2, \cdots a_n) = \begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-1} \\ s_1 & s_2 & s_3 & \cdots & s_n \\ s_2 & s_3 & s_4 & \cdots & s_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-2} \end{vmatrix}.$$

EXAMPLES.

1. Prove the result in Article 94, without reference to the theory of equations.

2. Prove that

$$\begin{vmatrix} 1 & a_2 + a_3 & a_2 a_3 \\ 1 & a_3 + a_1 & a_3 a_1 \\ 1 & a_1 + a_2 & a_1 a_2 \end{vmatrix} = -\zeta^{\frac{1}{2}}(a_1, a_2, a_3),$$

and give the corresponding expression for $\zeta^{\frac{1}{2}}(a_1, a_2, a_3, a_4)$.

3. Prove that

$$-3\zeta^{\frac{1}{2}}(a_1, a_2, a_3, a_4) = \left| \begin{array}{r} (a_4 - a_3)^3 (a_4 - a_2)^3 (a_4 - a_1)^3 \\ (a_3 - a_2)^3 (a_3 - a_1)^3 \\ (a_2 - a_1)^3 \end{array} \right|,$$

and give the general theorem.

4. Find the co-factor of $\zeta^{\frac{1}{2}}(a_1, \cdots a_n)\, \zeta^{\frac{1}{2}}(b_1, \cdots b_n)$

in $$\begin{vmatrix} (a_1 - b_1)^{n-1} & (a_1 - b_2)^{n-1} & \cdots & (a_1 - b_n)^{n-1} \\ (a_2 - b_1)^{n-1} & (a_2 - b_2)^{n-1} & \cdots & (a_2 - b_n)^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ (a_n - b_1)^{n-1} & (a_n - b_2)^{n-1} & \cdots & (a_n - b_n)^{n-1} \end{vmatrix}.$$

5. Find the co-factor of $\zeta^{\frac{1}{2}}(a_1, \cdots a_n)\, \zeta^{\frac{1}{2}}(b_1, \cdots b_n)$

in $$\begin{vmatrix} (a_1 - b_1)^{-1} & (a_1 - b_2)^{-1} & \cdots & (a_1 - b_n)^{-1} \\ (a_2 - b_1)^{-1} & (a_2 - b_2)^{-1} & \cdots & (a_2 - b_n)^{-1} \\ \cdot & \cdot & \cdot & \cdot \\ (a_n - b_1)^{-1} & (a_n - b_2)^{-1} & \cdots & (a_n - b_n)^{-1} \end{vmatrix}.$$

6. Show that

$$\begin{vmatrix} 1 & a_1 & a_1^3 \\ 1 & a_2 & a_2^3 \\ 1 & a_3 & a_3^3 \end{vmatrix} \div \zeta^{\frac{1}{2}}(a_1, a_2, a_3)$$

$$= \frac{a_1^3}{(a_1-a_2)(a_1-a_3)} + \frac{a_2^3}{(a_2-a_1)(a_2-a_3)} + \frac{a_3^3}{(a_3-a_1)(a_3-a_2)}$$

$$= a_1 + a_2 + a_3.$$

7. Show that

$$\begin{vmatrix} 1 & a_1 & \cdots & a_1^{n-2} & a_1^q \\ 1 & a_2 & \cdots & a_2^{n-2} & a_2^q \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_n & \cdots & a_n^{n-2} & a_n^q \end{vmatrix} \div \zeta^{\frac{1}{2}}(a_1, a_2, \cdots a_n)$$

$$= \frac{a_1^q}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)}$$

$$+ \frac{a_2^q}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)}$$

$$+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$+ \frac{a_r^q}{(a_r-a_1)(a_r-a_2)\cdots(a_r-a_{r-1})(a_r-a_{r+1})\cdots(a_r-a_n)}$$

$$+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$+ \frac{a_{n-1}}{(a_{n-1}-a_1)\cdots(a_{n-1}-a_{n-2})(a_{n-1}-a_n)}$$

$$+ \frac{a_n}{(a_n-a_1)\cdots(a_n-a_{n-2})(a_n-a_{n-1})}.$$

SUGGESTION. — Expand the given determinant dividend according to the elements of the last column.

8. By the aid of the preceding example show that

$$\frac{A(a_1^0,\, a_2,\, a_3^2,\, \cdots\, a_{n-1}^{n-2},\, a_n^q)}{\zeta^{\frac{1}{2}}(a_1,\, a_2,\, \cdots\, a_n)}$$

$$= \frac{a_n A(a_1^0,\, a_2,\, a_3^2,\, \cdots\, a_{n-1}^{n-2},\, a_n^{q-1})}{\zeta^{\frac{1}{2}}(a_1,\, a_2,\, \cdots\, a_n)}$$

$$+ \frac{A(a_1^0,\, a_2,\, a_3^2,\, \cdots\, a_{n-2}^{n-3},\, a_{n-1}^{q-1})}{\zeta^{\frac{1}{2}}(a_1,\, a_2,\, \cdots\, a_{n-1})}.$$

9. Show that

$$\frac{A(a_1^0,\, a_2,\, a_3^2,\, \cdots\, a_{n-1}^{n-2},\, a_n^q)}{\zeta^{\frac{1}{2}}(a_1,\, a_2,\, \cdots\, a_n)}$$

is the *complete* symmetric function of degree $(q-n+1)$ of the n quantities $a_1,\, a_2,\, \cdots\, a_n$.

10. As a special case of the above show that

$$\begin{vmatrix} 1 & x & x^5 \\ 1 & y & y^5 \\ 1 & z & z^5 \end{vmatrix} \div \zeta^{\frac{1}{2}}(x,\, y,\, z)$$

$$= x^3 + y^3 + z^3 + y^2z + yz^2 + x^2z + xz^2 + x^2y + xy^2 + xyz.$$

Continuants.

96. A determinant whose elements are all zero except those of the principal diagonal and the two adjacent minor diagonals, and in which the elements of one of these minor diagonals are all -1, is called a **continuant.** Thus

$$Q_n \equiv \begin{vmatrix} a_1 & -1 & 0 \cdots 0 & 0 \\ b_2 & a_2 & -1 \cdots 0 & 0 \\ 0 & b_3 & a_3 \cdots 0 & 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ 0 & 0 & 0 \cdots a_{n-1} & -1 \\ 0 & 0 & 0 \cdots b_n & a_n \end{vmatrix} \quad . \quad . \quad . \quad (1)$$

is a continuant of the nth order.

By an orderly change of columns into rows, we have

$$Q_n = \begin{vmatrix} a_1 & b_2 & 0 \cdots & 0 & 0 \\ -1 & a_2 & b_3 \cdots & 0 & 0 \\ 0 & -1 & a_3 \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \cdots & a_{n-1} & b_n \\ 0 & 0 & 0 \cdots & -1 & a_n \end{vmatrix}, \quad . \quad . \quad . \quad (2)$$

which shows that we may interchange the two minor diagonals.

Again, by reversing the order of both columns and rows, we may write

$$Q_n = \begin{vmatrix} a_n & -1 & 0 & \cdots 0 & 0 \\ b_n & a_{n-1} & -1 & \cdots 0 & 0 \\ 0 & b_{n-1} & a_{n-2} & \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots a_2 & -1 \\ 0 & 0 & 0 & \cdots b_2 & a_1 \end{vmatrix}; \quad . \quad . \quad (3)$$

that is, the order of the elements in the diagonals may be reversed.

The notation usually employed for continuants is as follows:

$$Q_n \equiv \begin{pmatrix} & b_2 & b_3 & \cdots & b_{n-1} & b_n & \\ a_1 & a_2 & a_3 & \cdots & & a_{n-1} & a_n \end{pmatrix}.$$

97. Expanding the second member of Equation (1) of the last article with reference to the elements of the last row, we have

$$Q_n = a_n \begin{pmatrix} & b_2 & b_3 & \cdots & b_{n-1} & \\ a_1 & a_2 & a_3 & \cdots & & a_{n-1} \end{pmatrix} + b_n \begin{pmatrix} & b_2 & b_3 & \cdots & b_{n-2} & \\ a_1 & a_2 & a_3 & \cdots & & a_{n-2} \end{pmatrix}$$

$$= a_n Q_{n-1} + b_n Q_{n-2}. \quad \ldots \ldots \ldots \ldots \quad (1)$$

This equation affords a convenient method of writing the expansion of a continuant in terms of minors co-axial with itself. Thus

$$\begin{pmatrix} & b_2 & b_3 & b_4 & \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix} = a_4 \begin{pmatrix} & b_2 & b_3 & \\ a_1 & a_2 & a_3 \end{pmatrix} + b_4 \begin{pmatrix} & b_2 & \\ a_1 & a_2 \end{pmatrix}$$

$$= a_4 a_3 \begin{pmatrix} & b_2 & \\ a_1 & a_2 \end{pmatrix} + a_4 b_3 (a_1) + b_4 \begin{pmatrix} & b_2 & \\ a_1 & a_2 \end{pmatrix}$$

$$= a_4 a_3 a_2 a_1 + a_4 a_3 b_2 + a_4 a_1 b_3 + a_2 a_1 b_4 + b_4 b_2.$$

98. If u_n is the number of terms in the expansion of the continuant of the order n, Equation (1) of the last article gives

$$u_n = u_{n-1} + u_{n-2}, \quad \ldots \ldots \ldots \ldots \quad (1)$$

a difference equation, the solution of which is

$$u_n = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} \quad \ldots \ldots \quad (2)$$

This is an integer for every integral value of n, as may be shown by the binomial theorem.

NOTE.—The reader who is not familiar with the Calculus of Finite Differences may deduce Equation (2) by the following somewhat arbitrary method:

Let Equation (1) be written

$$u_n = (\alpha + \beta)\, u_{n-1} - \alpha\beta u_{n-2}, \quad \ldots \ldots \quad (a)$$

α and β fulfilling the conditions

$$\alpha + \beta = 1 \text{ and } \alpha\beta = -1. \quad \ldots \ldots \quad (b)$$

From Equation (a) we have

$$u_n - \alpha u_{n-1} = \beta(u_{n-1} - \alpha u_{n-2}). \quad \ldots \ldots \quad (c)$$

This being true for all positive integral values of n, we may also write

$$\begin{aligned} u_{n-1} - \alpha u_{n-2} &= \beta(u_{n-2} - \alpha u_{n-3}), \\ u_{n-2} - \alpha u_{n-3} &= \beta(u_{n-3} - \alpha u_{n-4}), \\ \cdots \quad &\quad \cdots \\ u_3 - \alpha u_2 \quad &= \beta(u_2 - \alpha u_1). \end{aligned}$$

Hence Equation (c) may be written

$$u_n - \alpha u_{n-1} = (u_2 - \alpha u_1)\beta^{n-2}. \quad \ldots \ldots \quad (d)$$

Similarly,

$$u_n - \beta u_{n-1} = (u_2 - \beta u_1)\alpha^{n-2}. \quad \ldots \ldots \quad (e)$$

Eliminating u_{n-1} from Equations (d) and (e) we obtain

$$u_n = \frac{(u_2 - \alpha u_1)\beta^{n-1} - (u_2 - \beta u_1)\alpha^{n-1}}{\beta - \alpha}. \quad \ldots \quad (f)$$

Equations (b) give the values of α and β, and by inspection we have $u_2 = 2$ and $u_1 = 1$. Equation (f) thus becomes

$$u_n = \frac{\left(2 - \dfrac{1-\sqrt{5}}{2}\right)\left(\dfrac{1+\sqrt{5}}{2}\right)^{n-1} - \left(2 - \dfrac{1+\sqrt{5}}{2}\right)\left(\dfrac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}},$$

which is the same as Equation (2).

The value of u_n has been obtained by Professor Sylvester in the form

$$u_n = 1 + (n-1) + \frac{(n-2)(n-3)}{1\cdot 2}$$
$$+ \frac{(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3} + \cdots.$$

99. The determinant Q_n is equal to

$$q_n \equiv \begin{vmatrix} a_1 & \lambda_2 & 0 & \cdots & 0 & 0 \\ c_2 & a_2 & \lambda_3 & \cdots & 0 & 0 \\ 0 & c_3 & a_3 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_{n-1} & \lambda_n \\ 0 & 0 & 0 & \cdots & c_n & a_n \end{vmatrix}, \quad . \quad . \quad . \quad (1)$$

when $-\lambda_k c_k = b_k$.

For, developing in reference to the elements of the last row, we have

$$q_n = a_n q_{n-1} - \lambda_n c_n q_{n-2},$$

or $$q_n = a_n q_{n-1} + b_n q_{n-2}.$$

But
$$q_1 = Q_1 \text{ and } q_2 = Q_2;$$

also $$q_3 = a_3 Q_2 + b_3 Q_1 = Q_3; \text{ etc.}$$

Hence
$$q_n = Q_n.$$

Making
$$\lambda_k = 1,$$

and $$c_k = -b_k,$$

$$q_n = Q_n = \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 \\ -b_2 & a_2 & 1 & \cdots & 0 & 0 \\ 0 & -b_3 & a_3 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ 0 & 0 & 0 & \cdots & -b_n & a_n \end{vmatrix}; \quad . \quad . \quad (2)$$

that is, *the signs* $(-)$ *may be transferred from the elements of one minor diagonal to those of the other.*

100. Let us assume the equations

$$\left.\begin{array}{lll} b_1 & = a_1 x_1 & + x_2, \\ b_2 x_1 & = a_2 x_2 & + x_3, \\ b_3 x_2 & = a_3 x_3 & + x_4, \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \\ b_{n-1}x_{n-2} & = a_{n-1}x_{n-1} & + x_n, \\ b_n x_{n-1} & = a_n x_n & + x_{n+1}, \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \end{array}\right\} \quad . \quad . \quad . \quad (1)$$

From these equations we obtain successively

$$x_1 = \frac{b_1}{a_1 + \dfrac{x_2}{x^1}}, \qquad \frac{x_2}{x_1} = \frac{b_2}{a_2 + \dfrac{x_3}{x_2}}, \qquad \frac{x_3}{x_2} = \frac{b_3}{a_3 + \dfrac{x^4}{x_3}},$$

$$\cdots \qquad \frac{x_{n-1}}{x_{n-2}} = \frac{b_{n-1}}{a_{n-1} + \dfrac{x_n}{x_{n-1}}}, \quad \frac{x_n}{x_{n-1}} = \frac{b_n}{a_n + \dfrac{x_{n+1}}{x_n}}.$$

Hence

$$x_1 = \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots + \cfrac{b_{n-1}}{a_{n-1} + \cfrac{b_n}{a_n + \cfrac{x_{n+1}}{x_n}}}}}}.$$

Here x_1 is expressed as a continued fraction. The nth convergent of this fraction may be found by making x_{n+1} and all succeeding x's equal to zero.

We thus have for the nth convergent, writing it in the usual and more compact form,

$$x_1 = \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\cdots\frac{b_{n-1}}{+a_{n-1} +}\,\frac{b_n}{a_n}. \quad . \quad . \quad . \quad (2)$$

Now let the equations (1) be rewritten as follows, x_{n+1} and all succeeding x's being still assumed equal to zero:

$$\begin{aligned}
b_1 &= a_1x_1 + x_2,\\
0 &= -b_2x_1 + a_2x_2 + x_3,\\
0 &= \qquad -b_3x_2 + a_3x_3 + x_4,\\
&\cdots\cdots\cdots\cdots\\
0 &= \qquad\qquad -b_{n-1}x_{n-2} + a_{n-1}x_{n-1} + x_n,\\
0 &= \qquad\qquad\qquad -b_nx_{n-1} \quad + a_nx_n.
\end{aligned}$$

Solving these equations for x_1 by the method of Article 40 gives

$$x_1 = \frac{b_1 \begin{vmatrix} a_2 & 1 & 0 \cdots & 0 & 0 & 0 \\ -b_3 & a_3 & 1 \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 \cdots & -b_{n-1} & a_{n-1} & 1 \\ 0 & 0 & 0 \cdots & 0 & -b_n & a_n \end{vmatrix}}{\begin{vmatrix} a_1 & 1 & 0 & 0 \cdots & 0 & 0 & 0 \\ -b_2 & a_2 & 1 & 0 \cdots & 0 & 0 & 0 \\ 0 & -b_3 & a_3 & 1 \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 \cdots & -b_{n-1} & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 \cdots & 0 & -b_n & a_n \end{vmatrix}} \quad . \; . \; . \quad (3)$$

which we shall write in the abbreviated form

$$x_1 = \frac{P_n}{Q_n}.$$

We are thus able to express the nth convergent of a continued fraction in terms of two continuants. It is from this circumstance that the functions we are now considering have derived their name.

101. An inspection of Equation (3) of the last article shows that *the numerator of the determinant expression for* x_1 *is* b_1 *multiplied by the co-factor of* a_1 *in the denominator.* The reader familiar with the calculus will write (Eq. 1, Art. 39)

$$P_n = b_1 \frac{dQ_n}{da_1};$$

whence,

$$x_1 = b_1 \frac{\frac{dQ_n}{da_1}}{Q_n} = b_1 \frac{d(\log Q_n)}{da_1}.$$

102. Equation (2) of Article 63 gives at once

$$\delta \frac{d^2\delta}{da_{11}da_{nn}} = \frac{d\delta}{da_{11}} \cdot \frac{d\delta}{da_{nn}} - \frac{d\delta}{da_{n1}} \cdot \frac{d\delta}{da_{1n}}.$$

Taking Q_n for δ, then

$$\frac{d^2\delta}{da_{11}da_{nn}} = \frac{1}{b_1} P_{n-1}; \quad \frac{d\delta}{da_{11}} = \frac{1}{b_1} P_n;$$

$$\frac{d\delta}{da_{nn}} = Q_{n-1}; \quad \frac{d\delta}{da_{n1}} = 1 \text{ (Art. 29)};$$

$$\frac{d\delta}{da_{1n}} = (-1)^{n-1} b_2 b_3 \cdots b_n.$$

Hence

$$Q_n P_{n-1} - Q_{n-1} P_n = (-1)^n b_1 b_2 b_3 \cdots b_n.$$

103. The expression,

$$\chi_1 = \cfrac{\beta_1}{\alpha_1 + \cfrac{\beta_2}{\alpha_2 + \cfrac{\beta_3}{\alpha_3 + \cdots + \cfrac{\beta_{n-1}}{\alpha_{n-1} + \cfrac{\beta_n}{\alpha_n}}}}}$$

$$= \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \frac{\beta_3}{\alpha_3} + \cdots + \frac{\beta_{n-1}}{\alpha_{n-1}} + \frac{\beta_n}{\alpha_n},$$

is called an *ascending* continued fraction. Representing the nth convergent by

$$\frac{P'_n}{Q'_n},$$

we have

$$\frac{P'_1}{Q'_1}=\frac{\beta_1}{\alpha_1};\quad \frac{P'_2}{Q'_2}=\frac{\alpha_2\beta_1+\beta_2}{\alpha_1\alpha_2}=\frac{\alpha_2P'_1+\beta_2}{\alpha_2Q'_1};$$

$$\frac{P'_3}{Q'_3}=\frac{\alpha_3(\alpha_2\beta_1+\beta_2)+\beta_3}{\alpha_1\alpha_2\alpha_3}=\frac{\alpha_3P'_2+\beta_3}{\alpha_3Q'_2};$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\frac{P'_n}{Q'_n}=\frac{\alpha_nP'_{n-1}+\beta_n}{\alpha_nQ'_{n-1}}.$$

An inspection of these values of the successive convergents shows that

$$Q'_n=\alpha_1\alpha_2\alpha_3\cdots\alpha_n;$$

while P'_n is seen to be determined by the system

$$\begin{array}{lr}
P'_1 & =\beta_1, \\
-\alpha_2P'_1+P'_2 & =\beta_2, \\
\quad -\alpha_3P'_2+P'_3 & =\beta_3, \\
\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \\
\qquad\qquad -\alpha_{n-1}P'_{n-2}+P'_{n-1} & =\beta_{n-1}, \\
\qquad\qquad\qquad -\alpha_nP'_{n-1}+P'_n & =\beta_n.
\end{array}$$

The determinant of the coefficients of this system being unity (Art. 29), the solution is

$$P'_n = \begin{vmatrix} 1 & 0 & 0 \cdots & 0 & \beta_1 \\ -\alpha_2 & 1 & 0 \cdots & 0 & \beta_2 \\ 0 & -\alpha_3 & 1 \cdots & 0 & \beta_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \cdots & 1 & \beta_{n-1} \\ 0 & 0 & 0 \cdots & -\alpha_n & \beta_n \end{vmatrix} \quad . \quad . \quad . \quad . \quad (1)$$

By a simple transformation of the above determinant, and by writing Q'_n in determinant form, we have

$$\frac{P'_n}{Q'_n} = \frac{\begin{vmatrix} \beta_1 & -1 & 0 \cdots 0 & 0 \\ \beta_2 & \alpha_2 & -1 \cdots 0 & 0 \\ \beta_3 & 0 & \alpha_3 \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \beta_{n-1} & 0 & 0 \cdots \alpha_{n-1} & -1 \\ \beta_n & 0 & 0 \cdots 0 & \alpha_n \end{vmatrix}}{\begin{vmatrix} \alpha_1 & -1 & 0 \cdots 0 & 0 \\ 0 & \alpha_2 & -1 \cdots 0 & 0 \\ 0 & 0 & \alpha_3 \cdots 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \cdots \alpha_{n-1} & -1 \\ 0 & 0 & 0 \cdots 0 & \alpha_n \end{vmatrix}}, \quad . \quad . \quad (2)$$

in which the nth convergent of the ascending continued fraction χ_1 is expressed as the quotient of two determinants.

We now proceed to transform these determinants into continuants, by which means we shall be able to transform an ascending into a descending continued fraction.

104. Let us operate as follows upon both numerator and denominator of the second member of Equation (2) of the last article:

Multiply each element of the rth row, beginning with $r=n$, by β_{r-1}, and subtract from the product the corresponding element of the $(r-1)$th row multiplied by β_r, and continue this process through all the rows. The result is

$$\frac{P'_n}{Q'_n}=\frac{\begin{vmatrix} \beta_1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_2\beta_1+\beta_2 & -\beta_1 & \cdots & 0 & 0 & 0 \\ 0 & -\alpha_2\beta_3 & \alpha_3\beta_2+\beta_3 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \alpha_{n-2}\beta_{n-3}+\beta_{n-2} & -\beta_{n-3} & 0 \\ 0 & 0 & 0 & \cdots & -\alpha_{n-2}\beta_{n-1} & \alpha_{n-1}\beta_{n-2}+\beta_{n-1} & -\beta_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & -\alpha_{n-1}\beta_n & \alpha_n\beta_{n-1}+\beta_n \end{vmatrix}}{\begin{vmatrix} \alpha_1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -\alpha_1\beta_2 & \alpha_2\beta_1+\beta_2 & -\beta_1 & \cdots & 0 & 0 & 0 \\ 0 & -\alpha_2\beta_3 & \alpha_3\beta_2+\beta_3 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \alpha_{n-2}\beta_{n-3}+\beta_{n-2} & -\beta_{n-3} & 0 \\ 0 & 0 & 0 & \cdots & -\alpha_{n-2}\beta_{n-1} & \alpha_{n-1}\beta_{n-2}+\beta_{n-1} & -\beta_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & -\alpha_{n-1}\beta_n & \alpha_n\beta_{n-1}+\beta_n \end{vmatrix}}.$$

By Article 99 this equation becomes

$$\frac{P'_n}{Q'_n}=$$

$$\frac{\beta_1\begin{pmatrix} & -\alpha_2\beta_1\beta_3\cdots & & -\alpha_{n-2}\beta_{n-3}\beta_{n-1} & -\alpha_{n-1}\beta_{n-2}\beta_n & \\ \alpha_2\beta_1+\beta_2 & \alpha_3\beta_2+\beta_3 & \cdots & \alpha_{n-2}\beta_{n-3}+\beta_{n-2} & \alpha_{n-1}\beta_{n-2}+\beta_{n-1} & \alpha_n\beta_{n-1}+\beta_n\end{pmatrix}}{\begin{pmatrix} -\alpha_1\beta_2 & -\alpha_2\beta_1\beta_3 & & & -\alpha_{n-2}\beta_{n-3}\beta_{n-1} & -\alpha_{n-1}\beta_{n-2}\beta_n & \\ \alpha_1 & \alpha_2\beta_1+\beta_2 & \alpha_3\beta_2+\beta_3 & \cdots & \alpha_{n-2}\beta_{n-3}+\beta_{n-2} & \alpha_{n-1}\beta_{n-2}+\beta_{n-1} & \alpha_n\beta_{n-1}+\beta_n\end{pmatrix}},$$

which fulfills the condition stated in Article 101. Hence

$$\frac{P'_n}{Q'_n}=\frac{\beta_1}{\alpha_1-}\,\frac{\alpha_1\beta_2}{\alpha_2\beta_1+\beta_2-}\,\frac{\alpha_2\beta_1\beta_3}{\alpha_3\beta_2+\beta_3-}\cdots\frac{\alpha_{n-2}\beta_{n-3}\beta_{n-1}}{\alpha_{n-1}\beta_{n-2}+\beta_{n-1}-}\,\frac{\alpha_{n-1}\beta_{n-2}\beta_n}{\alpha_n\beta_{n-1}+\beta_n},$$

a formula by means of which we may transform any convergent of an ascending continued fraction into a descending continued fraction having the same number of quotients.

105. By means of the preceding article we may transform a given series into a descending continued fraction.

Let the series be

$$s=a_1+a_2+a_3+\cdots+a_n.$$

This may be written in the form of an ascending continued fraction, thus:

$$s=\frac{a_1}{1}+\frac{a_2}{1}+\frac{a_3}{1}+\ \cdots\ +\frac{a_n}{1}.$$

Transforming this into a descending continued fraction, we obtain

$$s = a_1 + a_2 + a_3 + \cdots + a_n$$

$$= \frac{a_1}{1-}\,\frac{a_2}{a_1+a_2-}\,\frac{a_1a_3}{a_2+a_3-}\cdots\frac{a_{n-3}a_{n-1}}{a_{n-2}+a_{n-1}-}\,\frac{a_{n-2}a_n}{a_{n-1}+a_n}.\ .\ (1)$$

If the given series is of the form

$$s' = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n},$$

it may, by means of (1), be converted into the continued fraction

$$s' = \frac{1}{a_1-}\,\frac{a_1^2}{a_1+a_2-}\,\frac{a_2^2}{a_2+a_3-}\cdots\frac{a_{n-2}^2}{a_{n-2}+a_{n-1}-}\,\frac{a_{n-1}^2}{a_{n-1}+a_n}.\ .\ .(2)$$

These formulæ are said to be due to Euler.

EXAMPLES.

Prove the following relations:

1. $\begin{pmatrix} & b_2 & b_3 & \cdots & b_n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$

$$= \begin{pmatrix} & b_2 & \cdots & b_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}\begin{pmatrix} & b_{k+2} & \cdots & b_n \\ a_{k+1} & a_{k+2} & \cdots & a_n \end{pmatrix}$$

$$+ a_{k+1}\begin{pmatrix} & b_2 & \cdots & b_{k-1} \\ a_1 & a_2 & \cdots & a_{k-1} \end{pmatrix}\begin{pmatrix} & b_{k+3} & \cdots & b_n \\ a_{k+2} & a_{k+3} & \cdots & a_n \end{pmatrix}.$$

2. $\begin{pmatrix} & b_1 & b_2 & b_3 & \\ 1 & a_1 & a_2 & a_3 & \cdots \end{pmatrix} = \begin{pmatrix} & b_2 & b_3 & \\ a_1 + b_1 & a_2 & a_3 & \cdots \end{pmatrix}.$

3. $$\begin{pmatrix} & b_2 & b_3 \cdots & b_n & \\ -a_1 & -a_2 & -a_3 \cdots & & -a_n \end{pmatrix}$$
$$= (-1)^n \begin{pmatrix} & b_2 & b_3 \cdots & b_n & \\ a_1 & a_2 & a_3 \cdots & & a_n \end{pmatrix}.$$

4. $$\begin{pmatrix} & 1 & 1 & 1 \cdots & 1 & \\ a_1x^{-1} & a_2x & a_3x^{-1} & a_4x \cdots & & a_nx^{(-1)^n} \end{pmatrix} = \begin{pmatrix} & 1 \cdots & 1 & \\ a_1 & a_2 \cdots & & a_n \end{pmatrix},$$

or
$$= x^{-1} \begin{pmatrix} & 1 \cdots & 1 & \\ a_1 & a_2 \cdots & & a_n \end{pmatrix}$$

according as n is even or odd.

5. Obtain the result in Article 99 by multiplying each row of the determinant (1) by $-\dfrac{1}{\lambda_{i+1}}$, and each column by $-\lambda_{i+1}$; i being, in each case, the number of the row or column multiplied.

6. Express the series

$$1 + 3 + 5 + 7 + \cdots + (2n-1)$$

in the form of a continued fraction.

7. Express the harmonic series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

in the form of a continued fraction.

8. Show that

$$\mu + \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \frac{b_n}{+ a_n} = \frac{\begin{pmatrix} & b_1 & b_2 \cdots & b_n & \\ \mu & a_1 & a_2 \cdots & & a_n \end{pmatrix}}{\begin{pmatrix} & b_2 & b_3 \cdots & b_n & \\ a_1 & a_2 & a_3 \cdots & & a_n \end{pmatrix}}.$$

9. Show that the periodic continued fraction

$$\overset{}{\underset{*}{\mu}} + \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\cdots\frac{b_3}{+a_3+}\,\frac{b_2}{a_2+}\,\frac{b_1}{\underset{*}{\mu}+}\,\mu+\cdots,$$

where the asterisks indicate the recurring period or *repetend,* is equal to

$$\left[\frac{\begin{pmatrix} & b_1 & b_2\cdots & b_3 & b_2 & b_1 & \\ \mu & a_1 & a_2\cdots & a_3 & a_1 & \mu \end{pmatrix}}{\begin{pmatrix} & b_1 & b_2\cdots & b_3 & b_2 & b_1 \\ a_1 & a_2 & a_3\cdots & a_3 & a_2 & a_1 \end{pmatrix}}\right]^{\frac{1}{2}}.$$

10. Show that

$$\frac{\begin{pmatrix} & 1 & 1\cdots & 1 \\ 1 & 1 & 1\cdots & 1 \end{pmatrix}_{n-1}}{\begin{pmatrix} & 1 & 1\cdots & 1 \\ 1 & 1 & 1\cdots & 1 \end{pmatrix}_{n}} = 2\frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}},$$

where n, or $n-1$, indicates the order of the continuant to which it is affixed.

CHAPTER VIII.

JACOBIANS, HESSIANS, AND WRONSKIANS.

In the applications of the calculus there frequently appear determinants whose elements are the differential coefficients of systems of functions. Most of the determinants originating in this manner come under one or another of the classes treated in the present chapter.

Jacobians.

106. Let $y_1, y_2, \cdots, y_n$ be n functions, each of n independent variables; thus:

$$\left.\begin{array}{l} y_1 = f_1(x_1, x_2, \cdots, x_n), \\ y_2 = f_2(x_1, x_2, \cdots, x_n), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_n = f_n(x_1, x_2, \cdots, x_n). \end{array}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

If now a determinant $|a_k^{(i)}|$ be formed in which $a_k^{(i)} \equiv \dfrac{\delta y_i}{\delta x_k}$, thus:

$$\begin{vmatrix} \frac{\delta y_1}{\delta x_1} & \frac{\delta y_2}{\delta x_1} & \cdots & \frac{\delta y_n}{\delta x_1} \\ \frac{\delta y_1}{\delta x_2} & \frac{\delta y_2}{\delta x_2} & \cdots & \frac{\delta y_n}{\delta x_2} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\delta y_1}{\delta x_n} & \frac{\delta y_2}{\delta x_n} & \cdots & \frac{\delta y_n}{\delta x_n} \end{vmatrix};$$

this determinant is called the **Jacobian*** of the functions $y_1, y_2, \cdots, y_n$ with respect to the variables $x_1, x_2, \cdots, x_n$. It is commonly denoted by

$$\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)}, \text{ or } J(y_1, y_2, \cdots, y_n).$$

NOTE. — If we have a single function of a single variable, as $y = f(x)$, the corresponding Jacobian is the differential coefficient $\frac{dy}{dx}$. This fact lies at the basis of the close analogy which we shall find to exist between Jacobians and ordinary differential coefficients, and suggests the first of the above notations. The second notation should be used only when there is no possible ambiguity in regard to the independent variables.

107. If the functions $y_1, y_2, \cdots, y_n$ are linear with respect to $x_1, x_2, \cdots, x_n$, thus:

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n, \\ y_2 &= a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_n &= a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n; \end{aligned}$$

* These functions were first studied by the Jewish mathematician Karl Gustav Jacob Jacobi (see Crelle's Journal, 1841), after whom they were named by Professor Sylvester.

then the corresponding Jacobian is the determinant of the coefficients of the linear functions, or

$$J(y_1, y_2, \cdots, y_n) = | a_{11}\, a_{22} \cdots a_{nn} |.$$

This follows directly from the definition of a Jacobian.

108. If the given functions $y_1, y_2, \cdots, y_n$ are independent, we may express $x_1, x_2, \cdots, x_n$ as so many functions of $y_1, y_2, \cdots, y_n$. In this case we have the following relation:

$$\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)} \cdot \frac{d(x_1, x_2, \cdots, x_n)}{d(y_1, y_2, \cdots, y_n)} = 1.$$

This may be shown by writing out each of the above Jacobians in determinant form, changing columns into rows in the first, and multiplying by rows. Thus, for $n=2$, we have

$$\begin{vmatrix} \frac{\delta y_1}{\delta x_1} & \frac{\delta y_1}{\delta x_2} \\ \frac{\delta y_2}{\delta x_1} & \frac{\delta y_2}{\delta x_2} \end{vmatrix} \cdot \begin{vmatrix} \frac{\delta x_1}{\delta y_1} & \frac{\delta x_2}{\delta y_1} \\ \frac{\delta x_1}{\delta y_2} & \frac{\delta x_2}{\delta y_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\delta y_1}{\delta x_1}\frac{\delta x_1}{\delta y_1} + \frac{\delta y_1}{\delta x_2}\frac{\delta x_2}{\delta y_1} & \frac{\delta y_1}{\delta x_1}\frac{\delta x_1}{\delta y_2} + \frac{\delta y_1}{\delta x_2}\frac{\delta x_2}{\delta y_2} \\ \frac{\delta y_2}{\delta x_1}\frac{\delta x_1}{\delta y_1} + \frac{\delta y_2}{\delta x_2}\frac{\delta x_2}{\delta y_1} & \frac{\delta y_2}{\delta x_1}\frac{\delta x_1}{\delta y_2} + \frac{\delta y_2}{\delta x_2}\frac{\delta x_2}{\delta y_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\delta y_1}{\delta y_1} & \frac{\delta y_1}{\delta y_2} \\ \frac{\delta y_2}{\delta y_1} & \frac{\delta y_2}{\delta y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

since y_1 and y_2 are independent.

109. If $y_1, y_2, \cdots, y_n$ are functions, each of $\sigma_1, \sigma_2, \cdots, \sigma_n$, and if these in turn are functions, each of $x_1, x_2, \cdots, x_n$, then

$$\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)} = \frac{d(y_1, y_2, \cdots, y_n)}{d(\sigma_1, \sigma_2, \cdots, \sigma_n)} \cdot \frac{d(\sigma_1, \sigma_2, \cdots, \sigma_n)}{d(x_1, x_2, \cdots, x_n)}.$$

This may be demonstrated by the method employed in the preceding article, remembering that

$$\frac{dy_i}{dx_k} = \frac{dy_i}{d\sigma_1}\,\frac{d\sigma_1}{dx_k} + \frac{dy_i}{d\sigma_2}\,\frac{d\sigma_2}{dx_k} + \cdots + \frac{dy_i}{d\sigma_n}\,\frac{d\sigma_n}{dx_k}.$$

110. If $y_1, y_2, \cdots, y_n$ are given only as implicit functions of $x_1, x_2, \cdots, x_n$, thus:

$$\begin{aligned} &\phi_1(y_1, \cdots, y_n, x_1, \cdots, x_n) = 0, \\ &\phi_2(y_1, \cdots, y_n, x_1, \cdots, x_n) = 0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\phi_n(y_1, \cdots, y_n, x_1, \cdots, x_n) = 0; \end{aligned} \qquad (1)$$

then

$$\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)} = (-1)^n \frac{\dfrac{d(\phi_1, \phi_2, \cdots, \phi_n)}{d(x_1, x_2, \cdots, x_n)}}{\dfrac{d(\phi_1, \phi_2, \cdots, \phi_n)}{d(y_1, y_2, \cdots, y_n)}}. \qquad (2)$$

Writing the above Jacobians in determinant form, changing rows to columns in the first member, clearing of fractions, and indicating the resulting product in the first member by P, gives

$$P \equiv \left| \frac{\delta\phi_i}{\delta y_1}\frac{\delta y_1}{\delta x_k} + \frac{\delta\phi_i}{\delta y_2}\frac{\delta y_2}{\delta x_k} + \cdots + \frac{\delta\phi_i}{\delta y_n}\frac{\delta y_n}{\delta x_k} \right| . \quad . \quad . \quad . \ (3)$$

Let us now form the total derivative of the function ϕ_i with respect to x_k. We have

$$\frac{d\phi_i}{dx_k} = \frac{\delta\phi_i}{\delta y_1}\frac{\delta y_1}{\delta x_k} + \frac{\delta\phi_i}{\delta y_2}\frac{\delta y_2}{\delta x_k} + \cdots + \frac{\delta\phi_i}{\delta y_n}\frac{\delta y_n}{\delta x_k}$$

$$+ \frac{\delta\phi_i}{\delta x_1}\frac{\delta x_1}{\delta x_k} + \cdots + \frac{\delta\phi_i}{\delta x_k} + \cdots + \frac{\delta\phi_i}{\delta x_n}\frac{\delta x_n}{\delta x_k} = 0.$$

The variables $x_1, x_2, \cdots, x_n$ being independent of one another, this last equation becomes

$$\frac{\delta\phi_i}{\delta y_1}\frac{\delta y_1}{\delta x_k} + \frac{\delta\phi_i}{\delta y_2}\frac{\delta y_2}{\delta x_k} + \cdots + \frac{\delta\phi_i}{\delta y_n}\frac{\delta y_n}{\delta x_k} = -\frac{\delta\phi_i}{\delta x_k}.$$

Substituting in Equation (3), there results

$$P = \left| -\frac{\delta\phi_i}{\delta x_k} \right| = (-1)^n \left| \frac{\delta\phi_i}{\delta x_k} \right|$$

$$= (-1)^n \frac{d(\phi_1, \phi_2, \cdots, \phi_n)}{d(x_1, x_2, \cdots, x_n)},$$

which proves the given theorem.

111. The reader will doubtless have noticed the analogy between the formulæ developed in the three preceding articles, and the following well-known formulæ of the differential calculus:

$$\frac{dy}{dx}\frac{dx}{dy}=1,$$

where $y=f(x)$;

$$\frac{dy}{dx}=\frac{dy}{d\sigma}\frac{d\sigma}{dx},$$

wnere $y=f(\sigma)$, and $\sigma=\phi(x)$; and

$$\frac{dy}{dx}=-\frac{\dfrac{d\phi}{dx}}{\dfrac{d\phi}{dy}},$$

where $\phi(y, x)=0$.

This analogy led Bertrand to devise a definition of a Jacobian which should itself be analogous to the definition of a simple differential coefficient. Bertrand's definition is as follows:

Referring to Equations (1) of Article 106, let

$$\left.\begin{array}{l}\Delta_1x_1,\ \Delta_1x_2,\ \cdots,\ \Delta_1x_n,\\ \Delta_2x_1,\ \Delta_2x_2,\ \cdots,\ \Delta_2x_n,\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \Delta_nx_1,\ \Delta_nx_2,\ \cdots,\ \Delta_nx_n,\end{array}\right\}; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be n distinct series of increments given to the n variables $x_1, x_2, \cdots, x$, and

$$\left.\begin{matrix} \Delta_1 y_1, \Delta_1 y_2, \cdots, \Delta_1 y_n, \\ \Delta_2 y_1, \Delta_2 y_2, \cdots, \Delta_2 y_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \Delta_n y_1, \Delta_n y_2, \cdots, \Delta_n y_n, \end{matrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the corresponding increments assumed by the functions. Then, just as the limiting value of the ratio $\frac{\Delta y}{\Delta x}$, that is $\frac{dy}{dx}$, is the differential coefficient of y with respect to x when $y=f(x)$, so is the Jacobian of $y_1, y_2, \cdots, y_n$, with respect to $x_1, x_2, \cdots, x_n$ the limiting ratio of the determinant of the systems of increments (2) and (1). Thus,

$$\frac{d\,(y_1, y_2, \cdots, y_n)}{d\,(x_1, x_2, \cdots, x_n)} = \frac{\begin{vmatrix} d_1 y_1 & d_2 y_1 & \cdots & d_n y_1 \\ d_1 y_2 & d_2 y_2 & \cdots & d_n y_2 \\ \cdot & \cdot & \cdot & \cdot \\ d_1 y_n & d_2 y_n & \cdots & d_n y_n \end{vmatrix}}{\begin{vmatrix} d_1 x_1 & d_1 x_2 & \cdots & d_1 x_n \\ d_2 x_1 & d_2 x_2 & \cdots & d_2 x_n \\ \cdot & \cdot & \cdot & \cdot \\ d_n x_1 & d_n x_2 & \cdots & d_n x_n \end{vmatrix}}.$$

This may be shown by clearing the above equation of fractions, and using the multiplication theorem, remembering that

$$d_k y_i = \frac{\delta y_i}{\delta x_1}\, d_k x_1 + \frac{\delta y_i}{\delta x_2}\, d_k x_2 + \cdots + \frac{\delta y_i}{\delta x_n}\, d_k x_n.$$

This definition may be used as a basis for the proof of each of the preceding formulæ. It is the natural basis of the arbitrary definition of a Jacobian first given.

112. *The Jacobian of a set of* n *functions,* $y_1, y_2, \cdots, y_n$, *each of the* n *independent variables* $x_1, x_2, \cdots, x_n$ *is equal to the product*

$$\frac{\delta\psi_1}{\delta x_1}\cdot\frac{\delta\psi_2}{\delta y_2}\cdots\frac{\delta\psi_n}{\delta y_n},$$

in which ψ_k *is a function of* $y_1, \cdots, y_{k-1}, x_k, \cdots, x_n$.

Letting $y_1 = \psi_1, (x_1, x_2, \cdots, x\ )$, we may obtain x_1 as a function of the variables $y_1, x_2, \cdots, x_n$.

Substituting this value of x, in the equation giving y_2, we have $y_2 = \psi_2\ (y_1, x_2, \cdots, x_n)$. This equation will make known x_2 as a function of $y_1, y_2, x_3, \cdots, x_n$, which, substituted in the equation for y_3 gives $y_3 = \psi_3\ (y_1, y_2, x_3, \cdots, x_n)$. We thus obtain the system

$$\begin{aligned}
y_1 &= \psi_1(x_1, x_2, x_3, \cdots, x_n),\\
y_2 &= \psi_2(y_1, x_2, x_3, \cdots, x_n),\\
&\cdots\cdots\cdots\cdots\\
y_k &= \psi_k(y_1, y_2, \cdots, y_{k-1}, x_k, \cdots, x_n),\\
&\cdots\cdots\cdots\cdots\\
y_n &= \psi_n(y_1, y_2, \cdots, y_{n-1}, x_n).
\end{aligned}$$

Forming the Jacobian of this system by the method of Article 110, remembering that

$$\frac{\delta\psi_i}{\delta x_k} = 0 \text{ for } k < i,$$

and that

$$\frac{\delta\psi_i}{\delta y_k} = 0 \text{ for } k > i,$$

we have

$$\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)} = (-1)^n \begin{vmatrix} -\frac{\delta\psi_1}{\delta x_1} & 0 & 0 & \cdots \\ -\frac{\delta\psi_1}{\delta x_2} & -\frac{\delta\psi_2}{\delta x_2} & 0 & \cdots \\ -\frac{\delta\psi_1}{\delta x_3} & -\frac{\delta\psi_2}{\delta x_3} & -\frac{\delta\psi_3}{\delta x_3} & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \div \begin{vmatrix} 1 & -\frac{\delta\psi_2}{\delta y_1} & -\frac{\delta\psi_3}{\delta y_1} & \cdots \\ 0 & 1 & -\frac{\delta\psi_3}{\delta y_2} & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

Hence

$$\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)} = \frac{\delta\psi_1}{\delta x_1} \cdot \frac{\delta\psi_2}{\delta x_2} \cdots \frac{\delta\psi_n}{\delta x_n}.$$

113. *If the Jacobian of a set of functions vanishes, the functions are not independent.*

For, if

$$\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)} = 0,$$

we have by the preceding article

$$\frac{\delta\psi_k}{\delta x_k} = 0,$$

in which k has some value between 1 and n.

Then ψ_k does not involve x_k, or

$$y_k = \psi_k(y_1, \cdots, y_{k-1}, x_{k+1}, \cdots, x_n).$$

Eliminating x_{k+1} between this equation and

$$y_{k+1} = \psi_{k+1}(y_1, \cdots, y_k, x_{k+1}, \cdots, x_n)$$

gives a result which may be written in the form

$$y_{k+1} = F_{k+1}(y_1, \cdots, y_k, x_{k+2}, \cdots, x_n);$$

that is, y_{k+1} does not involve x_{k+1}. In the same manner it may be shown that y_{k+2} does not involve $x_{k+2}, \cdots$, and finally that y_n does not involve x_n. Then

$$y_n = F(y_1, y_2, \cdots, y_{n-1}),$$

or the functions $y_1, y_2, \cdots, y_n$ are not independent.

114. We shall now demonstrate the converse of the theorem in the last article; that is:

If the functions $y_1, y_2, \cdots, y_n$ *are not independent, the Jacobian vanishes.*

Let

$$f(y_1, y_2, \cdots, y_n) = 0$$

be the equation connecting the given functions. Differentiating this function with respect to each of the variables $x_1, x_2, \cdots, x_n$ gives the consistent system,

$$\begin{array}{l} \dfrac{\delta f}{\delta y_1}\dfrac{\delta y_1}{\delta x_1} + \cdots + \dfrac{\delta f}{\delta y_n}\dfrac{\delta y_n}{\delta x_1} = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{\delta f}{\delta y_1}\dfrac{\delta y_1}{\delta x_n} + \cdots + \dfrac{\delta f}{\delta y_n}\dfrac{\delta y_n}{\delta x_n} = 0. \end{array}$$

Eliminating $\dfrac{\delta f}{\delta y_1}, \cdots, \dfrac{\delta f}{\delta y_n}$ from these equations gives

$$\begin{vmatrix} \dfrac{\delta y_1}{\delta x_1} & \cdots & \dfrac{\delta y_n}{\delta x_1} \\ \cdot & \cdot & \cdot \\ \dfrac{\delta y_1}{\delta x_n} & \cdots & \dfrac{\delta y_n}{\delta x_n} \end{vmatrix} \equiv J(y_1, y_2, \cdots, y_n) = 0.$$

115. If, in Article 109, the functions $\sigma_1, \sigma_2, \cdots, \sigma_n$ are linear, thus:

$$\left.\begin{array}{l} \sigma_1 = a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n, \\ \sigma_2 = a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \sigma_n = a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n; \end{array}\right\} \quad . \quad . \quad (1)$$

then

$$\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)} = \mu \cdot \frac{d(y_1, y_2, \cdots, y_n)}{d(\sigma_1, \sigma_2, \cdots, \sigma_n)},$$

in which μ is the determinant

$$a_{11}\ a_{22} \cdots a_{nn}\ |.$$

This follows at once from Articles 109 and 107. We have here a general case of the linear transformation of a set of functions, a special case of which has already been noticed in Article 54. The determinant μ is, as before, called the modulus of transformation. The contents of this article may now be summed up as follows:

If a set of functions be subjected to linear transformation, the Jacobian of the transformed functions is equal to the Jacobian of the original functions multiplied by the modulus of transformation.

For this reason the Jacobian is a *covariant* of the set of functions from which it is derived. (See Art. 128.)

Hessians.

116. The Jacobian of the first differential coefficients of a function of n variables, taken with respect to the several variables, is called the **Hessian** of the function.

Thus, if

$$u = f(\sigma_1, \sigma_2, \cdots, \sigma_n),$$

then is

$$\frac{d\left(\frac{\delta u}{\delta \sigma_1}, \frac{\delta u}{\delta \sigma_2}, \cdots, \frac{\delta u}{\delta \sigma_n}\right)}{d(\sigma_1, \sigma_2, \cdots, \sigma_n)} \equiv H(u)$$

the Hessian* of the function u.

Expressed in determinant form, we have

$$H(u) \equiv \begin{vmatrix} \frac{\delta^2 u}{\delta \sigma_1^2} & \frac{\delta^2 u}{\delta \sigma_2 \delta \sigma_1} \cdots & \frac{\delta^2 u}{\delta \sigma_n \delta \sigma_1} \\ \frac{\delta^2 u}{\delta \sigma_1 \delta \sigma_2} & \frac{\delta^2 u}{\delta \sigma_2^2} \cdots & \frac{\delta^2 u}{\delta \sigma_n \delta \sigma_2} \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \\ \frac{\delta^2 u}{\delta \sigma_1 \delta \sigma_n} & \frac{\delta^2 u}{\delta \sigma_2 \delta \sigma_n} \cdots & \frac{\delta^2 u}{\delta \sigma_n^2} \end{vmatrix};$$

from which it appears that the Hessian is an axi-symmetric determinant, since

$$\frac{\delta^2 u}{\delta \sigma_i \delta \sigma_k} = \frac{\delta^2 u}{\delta \sigma_k \delta \sigma_i}.$$

117. If the function u be transformed into a function u' by means of the linear substitutions indicated by the equations (1) of Article 115, then will

$$H(u') = \mu^2 H(u).$$

---

* These functions have been named after Professor Otto Hesse, by whom they had previously been called functional determinants.

That is to say:

If a function be subjected to linear transformation, the Hessian of the transformed function will equal the Hessian of the original function multiplied by the square of the modulus of transformation.

For, we have (Art. 115),

$$H(u') \equiv \frac{d\left(\frac{\delta u'}{\delta x_1}, \frac{\delta u'}{\delta x_2}, \cdots, \frac{\delta u'}{\delta x_n}\right)}{d(x_1, \quad x_2, \cdots, x_n)}$$

$$= \mu \frac{d\left(\frac{\delta u'}{\delta x_1}, \frac{\delta u'}{\delta x_2}, \cdots, \frac{\delta u'}{\delta x_n}\right)}{d(\sigma_1, \quad \sigma_2, \cdots, \sigma_n)}.$$

But, since

$$\frac{\delta^2 u'}{\delta \sigma_i \delta x_k} = \frac{\delta^2 u}{\delta \sigma_k \delta x_i},$$

the above equation may be written

$$H(u') = \mu \frac{d\left(\frac{\delta u}{\delta \sigma_1}, \frac{\delta u}{\delta \sigma_2}, \cdots, \frac{\delta u}{\delta \sigma_n}\right)}{d(x_1, \quad x_2, \cdots, x_n)}$$

$$= \mu^2 \frac{d\left(\frac{\delta u}{\delta \sigma_1}, \frac{\delta u}{\delta \sigma_2}, \cdots, \frac{\delta u}{\delta \sigma_n}\right)}{d(\sigma_1, \quad \sigma_2, \cdots, \sigma_n)}$$

$$= \mu^2 H(u).$$

It appears that the Hessian, like the Jacobian, is a covariant.

K Functions.

118. In connection with Jacobians may be noticed the function

$$\begin{vmatrix} u & u_1 & \cdots & u_n \\ \frac{\delta u}{\delta x_1} & \frac{\delta u_1}{\delta x_1} & \cdots & \frac{\delta u_n}{\delta x_1} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\delta u}{\delta x_n} & \frac{\delta u_1}{\delta x_n} & \cdots & \frac{\delta u_n}{\delta x_n} \end{vmatrix} \equiv K_x(u, u_1, \cdots, u_n).*$$

In the Jacobian in Article 106, let

$$y_1 = \frac{u_1}{u},\ y_2 = \frac{u_2}{u},\ \cdots,\ y_n = \frac{u_n}{u},$$

the functions having a common denominator u. Then

$$\frac{\delta y_i}{\delta x_k} = \frac{1}{u^2}\left(u\frac{\delta u_i}{\delta x_k} - u_i\frac{\delta u}{\delta x_k}\right),$$

and

$$u^{2n}\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)} = \begin{vmatrix} u\frac{\delta u_1}{\delta x_1} - u_1\frac{\delta u}{\delta x_1} & \cdots & u\frac{\delta u_n}{\delta x_1} - u_n\frac{\delta u}{\delta x_1} \\ \cdot & \cdot & \cdot \\ u\frac{\delta u_1}{\delta x_n} - u_1\frac{\delta u}{\delta x_n} & \cdots & u\frac{\delta u_n}{\delta x_n} - u_n\frac{\delta u}{\delta x_n} \end{vmatrix}$$

* These functions are called K functions or I functions indiscriminately; probably on account of their close relation to Jacobians, or J functions.

or (Eq. 1, Art. 29),

$$u^{2n+1}\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)}$$

$$\begin{vmatrix} u & u_1 & \cdots & u_n \\ 0 & u\dfrac{\delta u_1}{\delta x_1} - u_1\dfrac{\delta u}{\delta x_1} & \cdots & u\dfrac{\delta u_n}{\delta x_1} - u_n\dfrac{\delta u}{\delta x_1} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & u\dfrac{\delta u_1}{\delta x_n} - u_1\dfrac{\delta u}{\delta x_n} & \cdots & u\dfrac{\delta u_n}{\delta x_n} - u_n\dfrac{\delta u}{\delta x_n} \end{vmatrix}$$

$$\begin{vmatrix} u & u_1 & \cdots & u_n \\ u\dfrac{\delta u}{\delta x_1} & u\dfrac{\delta u_1}{\delta x_1} & \cdots & u\dfrac{\delta u_n}{\delta x_1} \\ \cdot & \cdot & \cdot & \cdot \\ u\dfrac{\delta u}{\delta x_n} & u\dfrac{\delta u_1}{\delta x_n} & \cdots & u\dfrac{\delta u_n}{\delta x_n} \end{vmatrix}. \qquad \text{(Art. 22.)}$$

Hence (Art. 20)

$$\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)} = \frac{1}{u^{n+1}} K_x(u, u_1, \cdots, u_n),$$

where $y_k = \dfrac{u_k}{u}$.

119. Referring to Article 106 again, if the functions $y_1, y_2, \cdots, y_n$ have a common factor, so that

$$y_1 = u_1 u,\ y_2 = u_2 u,\ \cdots,\ y_n = u_n u,$$

then

$$\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)} = 2u^n \frac{d(u_1, u_2, \cdots, u_n)}{d(x_1, x_2, \cdots, x_n)}$$
$$- u^{n-1} K_x(u, u_1, \cdots, u_n).$$

This may be shown by the method of the last article, remembering that

$$\frac{\delta y_i}{\delta x_k} = u\frac{\delta u_i}{\delta x_k} + u_i\frac{\delta u}{\delta x_k}.$$

Wronskians.

120. Let $y_1, y_2, \cdots, y_n$ be n functions of a single variable x.

If now a determinant $|a_k^{(i)}|$ be formed, in which

$$a_k^{(i)} \equiv \frac{d^{k-1}y_i}{dx^{k-1}};$$

thus:

$$\begin{vmatrix} y_1 & y_2 & \cdots y_n \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots \frac{dy_n}{dx} \\ \cdot & \cdot & \cdot \\ \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots \frac{d^{n-1}y_n}{dx^{n-1}} \end{vmatrix}$$

this determinant is called the **Wronskian** of the given functions with respect to x.*

We have the following abridged notations:

$$W_x(y_1, y_2, \cdots, y_n) \equiv \begin{vmatrix} y_1 & y_2 & \cdots y_n \\ y_1' & y_2' & \cdots y_n' \\ \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots y_n^{(n-1)} \end{vmatrix}_x; \quad . \quad . \quad . \quad (1)$$

* This name was given by Mr. Muir, after the Polish mathematician Höené Wronski (1778–1853), by whom the function was introduced in connection with the expansion-theorem which also bears his name. The Wronskian had been called simply the *determinant of the functions*, and written $D(y_1, y_2, \cdots, y_n)$.

in the second of which the order of the derivatives is indicated by superscripts.

121. Letting σ be any function of x, we may write arbitrarily,

$$\sigma^n = \begin{vmatrix} \sigma & \sigma' & \sigma'' & \sigma''' \cdots & \sigma^{(i)} & \cdots & \sigma^{(n-1)} \\ 0 & \sigma & 2\sigma' & 3\sigma'' \cdots & \binom{i}{1}\sigma^{(i-1)} & \cdots & \binom{n-1}{1}\sigma^{(n-2)} \\ 0 & 0 & \sigma & 3\sigma' \cdots & \binom{i}{2}\sigma^{(i-2)} & \cdots & \binom{n-1}{2}\sigma^{(n-3)} \\ 0 & 0 & 0 & \sigma \cdots & \binom{i}{3}\sigma^{(i-3)} & \cdots & \binom{n-1}{3}\sigma^{(n-4)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \cdots & 0 & \cdots & \sigma \end{vmatrix}, \quad (2)$$

in which $1, \binom{i}{1}, \binom{i}{2}, \binom{i}{3}, \cdots, 1$ are the coefficients of the expansion of $(a+b)^i$.

Multiplying together Equations (1) and (2), the second members by columns, we have

$$\sigma^n W(y_1, y_2, \cdots, y_n) =$$

$$\begin{vmatrix} y_1\sigma & y_2\sigma & y_3\sigma & \cdots \\ y_1\sigma'+y_1'\sigma & y_2\sigma'+y_2'\sigma & y_3\sigma'+y_3'\sigma & \cdots \\ y_1\sigma''+2y_1'\sigma'+y_1''\sigma & y_2\sigma''+2y_2'\sigma'+y_2''\sigma & y_3\sigma''+2y_3'\sigma'+y_3''\sigma & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

In accordance with the theorem of Leibnitz, the elements in the ith column of the determinant

in the second member are σy_i and its successive derivatives.

Hence

$$\sigma^n W(y_1, y_2, \cdots, y_n) = W(\sigma y_1, \sigma y_2, \cdots, \sigma y_n).$$

122. If we put $\sigma = \dfrac{1}{y_1}$, then

$$y_i\sigma' + y_i'\sigma = (y_i\sigma)' \text{ becomes } \left(\frac{y_i}{y_1}\right)',$$

which vanishes when $i = 1$.

The result in the last article thus gives

$$\frac{1}{y_1^n} W(y_1, y_2, \cdots, y_n) = W\left[\left(\frac{y_2}{y_1}\right)', \left(\frac{y_3}{y_1}\right)', \cdots, \left(\frac{y_n}{y_1}\right)\right], \quad (1)$$

the second member being of the order $(n-1)$.

But $\left(\dfrac{y_i}{y_1}\right)' = \dfrac{y_i'y_1 - y_iy_1'}{y_1^2} = \dfrac{1}{y_1^2} W(y_1, y_i).$

Letting $W(y_1, y_i) \equiv w_i$, Equation (1) becomes, by means of the last article,

$$W(y_1, y_2, \cdots, y_n) = \frac{1}{y_1^{n-2}} W(w_2, w_3, \cdots, w_n). \quad . \quad . \quad (2)$$

123. *If the functions are connected by a linear relation, as*

$$a_1y_1 + a_2y_2 + \cdots + a_ny_n = 0,$$

the Wronskian vanishes.

This may be shown by eliminating $a_1, a_2, \cdots, a_n$ from the above equation and its first $(n-1)$ derivatives.

124. The converse of the preceding theorem is also true. That is:

If the Wronskian of a set of functions vanishes, the functions are connected by a linear relation with constant coefficients.

We shall employ the method of induction. If, then,

$$W(y_1, y_2, \cdots, y_n) = 0,$$

we have, by Article 122,

$$W(w_2, w_3, \cdots, w_n) = 0.$$

The theorem being assumed true for $(n-1)$ functions, we have

$$a_2 w_2 + a_3 w_3 + \cdots + a_n w_n = 0;$$

or, dividing through by y_1^2,

$$a_2\left(\frac{y_2}{y_1}\right)' + a_3\left(\frac{y_3}{y_1}\right)' + \cdots + a_n\left(\frac{y_n}{y_1}\right)' = 0.$$

This gives

$$a_1 y_1 + a_2 y_2 + a_3 y_3 + \cdots + a_n y_n = 0,$$

in which a_1 is the constant of integration.

Hence, if the theorem is true for $(n-1)$ functions, it is also true for n functions. But, as is readily seen by inspection, it is true for *two* functions, and therefore generally.

EXAMPLES.

1. Find the Jacobian of the functions

$$y_1 = x_2^2 + 2a_1x_2x_3 + x_3^2,$$
$$y_2 = x_1^2 + 2a_2x_1x_3 + x_3^2,$$
$$y_3 = x_1^2 + 2a_3x_1x_2 + x_2^2.$$

2. Given the functions

$$y_1 = x_1(1 - x_2),$$
$$y_2 = x_1x_2(1 - x_3), \ldots,$$
$$y_{n-1} = x_1x_2 \ldots x_{n-1}(1 - x_n),$$
$$y_n = x_1x_2 \ldots x_n.$$

Show that,

$$\frac{d(y_1, y_2, \cdots, y_n)}{d(x_1, x_2, \cdots, x_n)} = x_1^{n-1}\, x_2^{n-2} \cdots x^2_{n-2}\, x_{n-1}.$$

3. If the functions are as follows:

$$y_1 = f_1(x_1),$$
$$y_2 = f_2(x_1, x_2), \cdots,$$
$$y_n = f_n(x_1, x_2, \cdots, x_n);$$

show that $$J = \frac{\delta y_1}{\delta x_1} \cdot \frac{\delta y_2}{\delta x_2} \cdots \frac{\delta y_n}{\delta x_n}.$$

4. Find the Jacobian of $y_1, y_2, \cdots, y_n$, being given

$$y_1 = (1 - x_1),$$
$$y_2 = x_1(1 - x_2),$$
$$y_3 = x_1 x_2 (1 - x_3), \cdots,$$
$$y_n = x_1 x_2 \cdots x_{n-1}(1 - x_n).$$

Ans. $(-1)^n x_1^{n-1} x_2^{n-2} \cdots x^2_{n-2} x_{n-1}$.

5. Find the Jacobian of $x_1, x_2, \cdots, x_n$ with respect to $\theta_1, \theta_2, \cdots, \theta_n$, being given

$$x_1 = \cos\theta_1,$$
$$x_2 = \sin\theta_1 \cos\theta_2,$$
$$x_3 = \sin\theta_1 \sin\theta_2 \cos\theta_3, \cdots,$$
$$x_n = \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1} \cos\theta_n.$$

6. Given $u = (r + a\cos\theta)^2$, $v = (r - b\sin\theta)^2$, in which $r = (x^2 + y^2)^{\frac{1}{2}}$, $\theta = \operatorname{arc\,tan}\dfrac{y}{x}$, to find $\dfrac{d(u, v)}{d(x, y)}$.

7. Given $y_1(x_1 - x_2) = 0$, $y_2(x_1^2 + x_1 x_2 + x_2^2) = 0$, to find $\dfrac{d(y_1, y_2)}{d(x_1, x_2)}$.

Ans. $3 y_1 y_2 \dfrac{x_1 + x_2}{x_1^3 - x_2^3}$.

8. Given $\dfrac{x\cos u}{v\cos y} = 0$, $\dfrac{u\sin x}{y\sin v} = 0$, to find $\dfrac{d(u, v)}{d(x, y)}$.

Ans. $-\dfrac{uv\cos u \sin v \sin(x + y)}{xy \sin x \cos y \sin(u + v)}$.

9. If $y_{k+1}, \cdots, y_n$ are independent of $x_1, \cdots, x_k$, or if $y_1, \cdots, y_k$ are independent of $x_{k+1}, \cdots, x_n$, show that

$$\frac{d(y_1, \cdots, y_k, y_{k+1}, \cdots, y_n)}{d(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n)} = \frac{d(y_1, \cdots, y_k)}{d(x_1, \cdots, x_k)} \cdot \frac{d(y_{k+1}, \cdots, y_n)}{d(x_{k+1}, \cdots, x_n)}.$$

10. The conditions of the preceding example still holding good, show that

$$\frac{d(y_1, \cdots, y_k, x_{k+1}, \cdots, x_n)}{d(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n)} = \frac{d(y_1, y_2, \cdots, y_k)}{d(x_1, x_2, \cdots, x_k)}.$$

11. If the functions $y_1, y_2, \ldots, y_n$ are independent, show, by means of the preceding example and Article 109, that

$$\frac{d(y_1, \ldots, y_n)}{d(x_1, \ldots, x_n)} \cdot \frac{d(x_1, \ldots, x_k)}{d(y_1, \ldots, y_k)} = \frac{d(y_{k+1}, \ldots, y_n)}{d(x_{k+1}, \ldots, x_n)}.$$

12. Prove the formula of Article 109 by means of Bertrand's definition of a Jacobian.

13. Find the Jacobian of the functions,

$$y_1 = (x_1 - x_2)(x_2 + x_3),$$
$$y_2 = (x_1 + x_2)(x_2 - x_3),$$
$$y_3 = x_2(x_1 - x_3),$$

and thus show that the functions are not independent.

14. Find the Jacobian of $u = x(y+z)$, $v = y(x+z)$, $w = z(x-y)$, and thus show that u, v, w are not independent.

15. If $x = r\cos\theta$, $y = r\sin\theta$, in which $r = au$, $\theta = bv$, find $\dfrac{d(x, y)}{d(u, v)}$. *Ans. abr.*

16. Given, $u = x + y$, $v = xy$, in which

$$x = x'\cos\alpha - y'\sin\alpha, \; y = x'\sin\alpha + y'\cos\alpha,$$

to find $\dfrac{d(u, v)}{d(x, y)}$.

17. Find the Hessian of the function,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0.$$

18. Find the Hessian of the function given in the last example upon introducing the substitutions,

$$x = l_1x' + m_1y' + n_1z',$$
$$y = l_2x' + m_2y' + n_2z',$$
$$z = l_3x' + m_3y' + n_3z'.$$

19. Find the Hessian of the function,

$$a_{30}x_1^3 + 3\,a_{21}x_1^2x_2 + 3\,a_{12}x_1x_2^2 + a_{03}x_2^3.$$

$$Ans.\ 36\left\{\begin{vmatrix} a_{30} & a_{21} \\ a_{21} & a_{12}\end{vmatrix} x_1^2 + \begin{vmatrix} a_{30} & a_{12} \\ a_{21} & a_{03}\end{vmatrix} x_1x_2 + \begin{vmatrix} a_{21} & a_{12} \\ a_{12} & a_{03}\end{vmatrix} x_2^2\right\}.$$

20. Show that, if a homogeneous relation exists among the functions $u, u_1, \ldots, u_n$, then is

$$K(u, u_1, \ldots, u_n) = 0.$$

21. If $u_i = \frac{v_i}{t}$, show that

$$K(u, u_1, \ldots, u_n) = \frac{1}{t^{n+1}} K(v, v_1, \ldots, v_n).$$

22. Prove that, if $y_1, y_2, \ldots, y_n$ are functions of σ, and σ a function of x, then is

$$W_x(y_1, y_2, \ldots, y_n) = \left(\frac{d\sigma}{dx}\right)^{\frac{n(n-1)}{2}} W_\sigma(y_1, y_2, \ldots, y_n).$$

23. Show that the total differential of a Wronskian with respect to its independent variable is obtained by differentiating the elements of the last row.

24. Prove the formula,

$$W(y_1, y_2, \cdots, y_n)$$

$$= \frac{1}{[W(y_1, y_2)]^{n-3}} W\{W(y_1, y_2, y_3), W(y_1, y_2, y_4), \cdots, W(y_1, y_2, y_n)\}.$$

25. Generalize the preceding example.

26. Find the Wronskian of the functions $x^n + a$, $x^n + b$, $x^n + c$, and thus show that these functions are connected by a linear relation. Also, find this linear relation.

27. Show that a linear relation exists among the functions, $\sin x$, $e^{x\sqrt{-1}}$, $\cos x$.

CHAPTER IX.

LINEAR TRANSFORMATIONS.

The theory of linear transformations, in its application to quantics, and particularly to quadrics, is of the greatest importance in connection with the study of modern geometry.

The student wishing an extended course in this subject should read Salmon's *Modern Higher Algebra*, or perhaps better still, Clebsch's *Vorlesungen über Geometrie*, where it is presented in its true geometrical relations.

The greater number of the articles in the present chapter are capable of direct geometrical interpretation; though that interpretation is only hinted at in two or three instances, being outside the scope of the present work.

125. A **quantic** is a homogeneous function of any number of variables and of any degree.

The number of variables is indicated by some one of the adjectives: binary, ternary, quaternary, etc.; while the function is called a quadric, a cubic, a quartic, etc., according to its degree. In

general, the quantic involving n variables and of the mth degree, is called an $n$ary $m$ic. Thus, as an example of a binary cubic, we have

$$a_{30}x_1^3 + 3a_{21}x_1^2x_2 + 3a_{12}x_1x_2^2 + a_{03}x_2^3.$$

This is commonly denoted by the symbol

$$(a_{30}, a_{21}, a_{12}, a_{03} \between x_1, x_2),$$

when the numerical coefficients are those of the expansion of $(x_1+x_2)^3$. When the terms are not affected by numerical coefficients, the notation is

$$(a_{30}, a_{21}, a_{12}, a_{03} \between x_1, x_2) \equiv a_{30}x_1^3 + a_{21}x_1^2x_2 + a_{12}x_1x_2^2 + a_{03}x_2^3.$$

The same notation is used for quantics in general.

126. The **discriminant** of a quantic is the eliminant of its first derivatives taken with respect to its several variables. Thus, if

$$q \equiv a_{30}x_1^3 + 3a_{21}x_1^2x_2 + 3a_{12}x_1x_2^2 + a_{03}x_2^3,$$

we have

$$q_1 \equiv \frac{\delta q}{\delta x_1} = 3a_{30}x_1^2 + 6a_{21}x_1x_2 + 3a_{12}x_2^2,$$

$$q_2 \equiv \frac{\delta q}{\delta x_2} = 3a_{21}x_1^2 + 6a_{12}x_1x_2 + 3a_{03}x_2^2;$$

and the discriminant is the eliminant of q_1 and q_2, which is (Art. 47)

$$\delta \equiv \begin{vmatrix} 0 & 3a_{30} & 6a_{21} & 3a_{12} \\ 3a_{30} & 6a_{21} & 3a_{12} & 0 \\ 0 & 3a_{21} & 6a_{12} & 3a_{03} \\ 3a_{21} & 6a_{12} & 3a_{03} & 0 \end{vmatrix}.$$

127. An **invariant** is a function of the coefficients of a quantic which is not affected by linear transformation of the quantic, excepting that it is multiplied by a power of the modulus of transformation.

Thus, if the quantic q in the preceding article be transformed to q' by the linear substitutions

$$x_1 = b_{11}u_1 + b_{12}u_2, \quad x_2 = b_{21}u_1 + b_{22}u_2,$$

and if the discriminant δ' of the quantic q' be formed, we shall have

$$\delta = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}^3 \cdot \delta.$$

Hence δ is an invariant of q.

128. A **covariant** of a quantic is a function involving both the coefficients and the variables of the quantic and so related to it that, when the quantic is subjected to linear transformation, the same function of the new coefficients and variables shall equal the original function multiplied by a power of the modulus of transformation.

We have already noted the Hessian as a covariant of the function from which it is derived. (Art. 117.)

As an example of a covariant of a set of functions we have the Jacobian. (Art. 115.)

It follows at once from the definitions that *every invariant of a covariant is an invariant of the original quantic.*

129. The *quadric* in n variables, $x_1, x_2, \cdots, x_n$, is usually denoted by

$$q = \Sigma\Sigma a_{ik}x_i x_k,$$

in which the coefficient of x_i^2 is a_{ii} and that of $x_i x_k$ is $2a_{ik}$, a_{ik} *being identical with* a_{ki}. We have

$$q_1 \equiv \frac{1}{2}\frac{\delta q}{\delta x_1} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n,$$

$$q_2 \equiv \frac{1}{2}\frac{\delta q}{\delta x_2} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$q_n \equiv \frac{1}{2}\frac{\delta q}{\delta x_n} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n.$$

Hence, the discriminant of the general quadric is the symmetrical determinant,

$$\delta \equiv \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix}.$$

It may be observed that

$$H(q) = 2^n \cdot \delta;$$

that is:

The discriminant and the Hessian of a quadric are both invariants. Upon transformation of the quadric by linear substitution, each is multiplied by the square of the modulus. (Art. 117.)

130. *If a binary quantic contains a square factor, the discriminant vanishes.*

Every binary quantic containing a square factor may be written in the form

$$q \equiv (ax + by)^2 \cdot \phi(x, y).$$

Now, the derivatives $\frac{\delta q}{\delta x}$ and $\frac{\delta q}{\delta y}$ contain a common factor, $(ax + by)$. Therefore the eliminant of these derivatives, which is the discriminant of q, must vanish. Hence the theorem.

It follows that

The discriminant of a binary quantic is an invariant.

131. We shall now show that

If any quadric is resolvable into two homogeneous linear factors, the discriminant vanishes.

Let $q = \psi \cdot \psi'$ be the quadric, where

$$\psi \equiv a_1x_1 + a_2x_2 + \cdots + a_nx_n,$$

$$\psi' \equiv a_1'x_1 + a_2'x_2 + \cdots + a_n'x_n.$$

Then

$$q_1 = a_1\psi' + a'_1\psi, \quad q_2 = a_2\psi' + a'_2\psi, \quad \cdots,$$
$$q_n = a_n\psi' + a'_n\psi.$$

Hence, $q_1, q_2, \cdots, q_n$ have for common roots, the roots of the simultaneous equations

$$\psi = 0 \text{ and } \psi' = 0,$$

and the eliminant of the equations,

$$q_1 = 0,\ q_2 = 0,\ \cdots,\ q_n = 0,$$

vanishes. This eliminant being the discriminant of q, the theorem is established.

In Article 50 we have already treated a special case of this theorem.

NOTE. — The reader will perhaps meet with difficulty in applying this theorem to some expression of the form

$$q \equiv aa'x^2 + (ab' + a'b)xy + bb'y^2,$$

which is obviously the same as

$$(ax + by)(a'x + b'y) = \psi \cdot \psi',$$

no matter what relation may exist among a, b, a', and b'.

Placing the derivatives of q equal to zero, we have

$$a\psi' + a'\psi = 0, \quad b\psi' + b'\psi = 0.$$

If these equations are consistent, that is, if the discriminant vanishes, we have either

$$\psi = 0,\ \psi' = 0;\ \text{or} \begin{vmatrix} a & a' \\ b & b' \end{vmatrix} = 0.$$

But these two conditions are identical, and equivalent to

$$\frac{a'}{a} = \frac{b'}{b} = \lambda,$$

where λ is some constant. Hence

$$\psi\psi' = \lambda\psi^2;$$

or, the vanishing of the discriminant of a binary quadric indicates that it contains a square factor, as has been shown in the preceding article. If, however, the discriminant does not vanish, we cannot have $\psi=0$, $\psi'=0$ *simultaneously*, unless $x=y=0$. Hence, *when the discriminant of a binary quadric does not vanish the quadric expresses two independent relations between the variables,* which are real or imaginary according to the sign of the discriminant. (See Art. 49.)

132. Given the two ternary quadrics

$$q \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{23}x_2x_3 + 2a_{13}x_1x_3 + 2a_{12}x_1x_2,$$
$$q' \equiv a'_{11}x_1^2 + a'_{22}x_2^2 + a'_{33}x_3^2 + 2a'_{23}x_2x_3 + 2a'_{13}x_1x_3 + 2a'_{12}x_1x_2.$$

Let it be required to find the (three) values of k for which the ternary quadric

$$kq + q'$$

shall be resolvable into linear factors.

NOTE. — The student familiar with the methods of analytical geometry will interpret this as follows: Given the two conics $q = 0$ and $q' = 0$, to find the values of k for which $kq + q' = 0$, represents one of the three pairs of right lines passing through the four points of intersection of the given conics.

The given condition requires that the discriminant of $(kq + q')$ shall vanish. That is:

$$\begin{vmatrix} ka_{11} + a'_{11} & ka_{12} + a'_{12} & ka_{13} + a'_{13} \\ ka_{21} + a'_{21} & ka_{22} + a'_{22} & ka_{23} + a'_{23} \\ ka_{31} + a'_{31} & ka_{32} + a'_{32} & ka_{33} + a'_{33} \end{vmatrix} = 0.$$

Expanding this determinant gives

$$\Delta k^3 + \Theta k^2 + \Theta' k + \Delta' = 0,$$

in which

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\Theta \equiv A_{11}a'_{11} + A_{22}a'_{22} + A_{33}a'_{33} + 2\,A_{23}a'_{23} + 2\,A_{13}a'_{13} + 2\,A_{12}a'_{12},$$
$$\Theta' \equiv A'_{11}a_{11} + A'_{22}a_{22} + A'_{33}a_{33} + 2\,A'_{23}a_{23} + 2\,A'_{13}a_{13} + 2\,A'_{12}a_{12},$$

$$\Delta' \equiv \begin{vmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{vmatrix};$$

A_{11}, ... and A'_{11}, ... being co-factors of the elements of Δ and Δ' respectively.

The above equation in k being of the third degree shows that there are three values of this parameter which fulfill the required condition, at least one of which must be real.

The coefficients Δ, Θ, Θ', and Δ' are invariants, for k is manifestly not altered by linear transformation of q and q', and the ratios of these coefficients must therefore remain unaltered by any such transformation.

133. If any function of the variables $x_1, x_2, \ldots, x_n$ be transformed into a function of $u_1, u_2, \ldots, u_n$ by means of the linear substitutions,

$$\left.\begin{array}{l} x_1 = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n, \\ x_2 = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n; \end{array}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

the coefficients a_{ik} being so chosen that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = u_1^2 + u_2^2 + \cdots + u_n^2, \quad . \quad . \quad (2)$$

the transformation is said to be **orthogonal.**

NOTE. — The transformation, in analytical geometry, from one set of axes to another, without changing the origin, is orthogonal.

Substituting the values of $x_1, \ldots, x_n$ given by the equations (1) in Equation (2), and equating coefficients, gives

$$\left.\begin{array}{l} a_{1i}^2 \;\; + a_{2i}^2 \;\; + \cdots + a_{ni}^2 \;\;\; = 1, \\ a_{1i}a_{1k} + a_{2i}a_{2k} + \cdots + a_{ni}a_{nk} = 0; \end{array}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

where $i = 1, 2, \ldots, n$, successively,

and $\quad k = 1, 2, \ldots, i-1, i+1, \ldots, n.$

Multiplying each of Equations (1) by the coefficient of u_i in its second member, and adding, we obtain, by the aid of Equations (3),

$$\left.\begin{array}{l} u_1 = a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n, \\ u_2 = a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_n = a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

Hence, using the method employed to obtain Equations (3), we also have

$$\left.\begin{aligned} a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2 &= 1, \\ a_{i1}a_{k1} + a_{i2}a_{k2} + \cdots + a_{in}a_{kn} &= 0; \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (5)$$

i and k being used as before.

Squaring the determinant,

$$\mu = \begin{vmatrix} a_{11} \ldots a_{1n} \\ . \quad . \quad . \\ a_{n1} \ldots a_{nn} \end{vmatrix},$$

all the elements of the square vanish except those on the principal diagonal, and each of these is unity (Eqs. 3 or 5); which proves that

The square of the modulus of an orthogonal transformation is unity.

134. Equations (3), or (5), assign $\frac{1}{2}n(n+1)$ conditions to be fulfilled by the n^2 coefficients of an orthogonal transformation, leaving $\frac{1}{2}n(n-1)$ conditions unassigned. We may therefore express these n^2 coefficients in terms of any $\frac{1}{2}n(n-1)$ quantities arbitrarily chosen, such as

$$\begin{array}{l} b_{12},\, b_{13},\, \ldots,\, b_{1n}, \\ \qquad b_{23},\, \ldots,\, b_{2n}, \\ \qquad\quad . \quad . \quad . \quad . \\ \qquad\qquad b_{n-1,\,n}. \end{array}$$

Letting $b_{ii}=1$, and $b_{ik}=-b_{ki}$, we assume two systems of linear substitutions, thus:

$$\left.\begin{array}{l} x_1 = b_{11}\sigma_1 + b_{12}\sigma_2 + \cdots + b_{1n}\sigma_n, \\ x_2 = b_{21}\sigma_1 + b_{22}\tau_2 + \cdots + b_{2n}\sigma_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n = b_{n1}\sigma_1 + b_{n2}\tau_2 + \cdots + b_{nn}\sigma_n\,; \end{array}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\left.\begin{array}{l} u_1 = b_{11}\tau_1 + b_{21}\tau_2 + \cdots + b_{n1}\tau_n, \\ u_2 = b_{12}\sigma_1 + b_{22}\tau_2 + \cdots + b_{n2}\sigma_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_n = b_{1n}\sigma_1 + b_{2n}\tau_2 + \cdots + b_{nn}\sigma_n. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

Adding the corresponding equations of these two systems gives

$$x_1 + u_1 = 2\,\sigma_1,\ x_2 + u_2 = 2\,\sigma_2,\ \cdots,\ x_n + u_n = 2\,\sigma_n. \quad . \quad . \quad . \quad (3)$$

Solving Equations (1) for σ_1, σ_2, $\cdots$, σ_n, we obtain

$$\left.\begin{array}{l} B\sigma_1 = B_{11}x_1 + B_{21}x_2 + \cdots + B_{n1}x_n\,, \\ B\sigma_2 = B_{12}x_1 + B_{22}x_2 + \cdots + B_{n2}x_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ B\sigma_n = B_{1n}x_1 + B_{2n}x_2 + \cdots + B_{nn}x_n; \end{array}\right\} \quad . \quad . \quad . \quad (4)$$

where B is the (skew) determinant of the given equations, and B_{ik} the co-factor of b_{ik} in B.

These last equations become, by means of Equations (3),

$$\left.\begin{array}{l} Bu_1 = (2B_{11} - B)x_1 + 2B_{21}x_2 + \cdots + 2B_{n1}x_n, \\ Bu_2 = 2B_{12}x_1 + (2B_{22} - B)x_2 + \cdots + 2B_{n2}x_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Bu_n = 2B_{1n}x_1 + 2B_{2n}x_2 + \cdots + (2B_{nn} - B)x_n. \end{array}\right\} \quad \ldots \ldots (5)$$

Solving Equations (2) and proceeding in exactly the same manner as above, we also obtain

$$\left.\begin{array}{l} Bx_1 = (2B_{11} - B)u_1 + 2B_{12}u_2 + \cdots + 2B_{1n}u_n, \\ Bx_2 = 2B_{21}u_1 + (2B_{22} - B)u_2 + \cdots + 2B_{2n}u_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Bx_n = 2B_{n1}u_1 + 2B_{n2}u_2 + \cdots + (2B_{nn} - B)u_n. \end{array}\right\} \quad \ldots \ldots (6)$$

If now we write

$$\frac{2B_{ii} - B}{B} \equiv a_{ii} \text{ and } \frac{2B_{ik}}{B} \equiv a_{ik},$$

the systems (6) and (5) become respectively the same as the systems (1) and (4) of the preceding article which we have shown to be connected by an orthogonal relation.

To obtain the coefficients of a binary orthogonal transformation then, let us assume

$$B = \begin{vmatrix} 1 & \lambda \\ -\lambda & 1 \end{vmatrix} = 1 + \lambda^2.$$

Here

$$B_{12} = \lambda,\ B_{21} = -\lambda,\ B_{11} = B_{22} = 1,$$

and

$$a_{11} = \frac{1-\lambda^2}{1+\lambda^2},\ a_{12} = \frac{2\lambda}{1+\lambda^2},\ a_{21} = \frac{-2\lambda}{1+\lambda^2},\ a_{22} = \frac{1-\lambda^2}{1+\lambda^2};$$

which gives for the required linear substitutions:

$$x_1 = \frac{1-\lambda^2}{1+\lambda^2}u_1 + \frac{2\lambda}{1+\lambda^2}u_2,$$

$$x_2 = \frac{-2\lambda}{1+\lambda^2}u_1 + \frac{1-\lambda^2}{1+\lambda^2}u_2;$$

in which λ may be chosen arbitrarily.

For the ternary orthogonal transformation we have, λ, μ, and ν being arbitrary,

$$B = \begin{vmatrix} 1 & \nu & -\mu \\ -\nu & 1 & \lambda \\ \mu & -\lambda & 1 \end{vmatrix},$$

so that

$$\begin{aligned} B_{11} &= 1+\lambda^2, & B_{12} &= \nu+\lambda\mu, & B_{13} &= \mu+\lambda\nu, \\ B_{21} &= -\nu+\lambda\mu, & B_{22} &= 1+\mu^2, & B_{23} &= \lambda+\mu\lambda, \\ B_{31} &= \mu+\lambda\nu, & B_{32} &= -\lambda+\mu\nu, & B_{33} &= 1+\nu^2. \end{aligned}$$

Hence the coefficients are, in order,

$$\begin{array}{lll} \dfrac{1+\lambda^2-\mu^2-\nu^2}{B}, & 2\dfrac{\nu+\lambda\mu}{B}, & 2\dfrac{-\mu+\lambda\nu}{B}, \\[2ex] 2\dfrac{-\nu+\lambda\mu}{B}, & \dfrac{1-\lambda^2+\mu^2-\nu^2}{B}, & 2\dfrac{\lambda+\mu\nu}{B}, \\[2ex] 2\dfrac{\mu+\lambda\nu}{B}, & 2\dfrac{-\lambda+\mu\nu}{B}, & \dfrac{1-\lambda^2-\mu^2+\nu^2}{B}. \end{array}$$

The coefficients for orthogonal transformations of higher orders may be found in the same manner.

135. Variables which are subjected to the same linear transformation are said to be **cogredient.**

Thus, if a function of $x_1, \ldots, x$ is transformed by the substitutions

$$x_1 = a_{11}u_1 + \cdots + a_{1n}u_n,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_n = a_{n1}u_1 + \cdots + a_{nn}u_n,$$

and a function of $x'_1, \ldots, x'_n$ by

$$x'_1 = a_{11}u'_1 + \cdots + a_{1n}u'_n,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x'_n = a_{n1}u'_1 + \cdots + a_{nn}u'_n,$$

then $x_1, \ldots, x_n$ are cogredient with $x'_1, \ldots, x'_n$.

NOTE. — When, in analytical geometry, we transform to new axes, the co-ordinates (x, y, z), (x', y', z'), $\cdots$, expressing different points, are cogredient.

Let us transform the function $\phi(x_1, x_2)$ by means of the substitutions

$$x_1 = x_1 + jx'_1,$$
$$x_2 = x_2 + jx'_2,$$

where x_1, x_2 are cogredient with x'_1, x'_2.

Developing the new function thus obtained by means of Taylor's theorem, we have

$$\phi(x_1 + jx'_1, x_2 + jx'_2) = \phi + j\left(x'_1\frac{\delta\phi}{\delta x_1} + x'_2\frac{\delta\phi}{\delta x_2}\right)$$

$$+ \frac{1}{1\cdot 2}j^2\left(x'^2_1\frac{\delta^2\phi}{\delta x_1^2} + 2x'_1x'_2\frac{\delta^2\phi}{\delta x_1\delta x_2} + x'^2_2\frac{\delta^2\phi}{\delta x_2^2}\right)$$

$$+ \frac{1}{1\cdot 2\cdot 3}j^3\left(x'^3_1\frac{\delta^3\phi}{\delta x_1^3} + 3x'^2_1x'_2\frac{\delta^3\phi}{\delta x_1^2\delta x_2} + 3x'_1x'^2_2\frac{\delta^3\phi}{\delta x_1\delta x_2^2} + x'^3_2\frac{\delta^3\phi}{\delta x_2^3}\right)$$

$$+ \cdots.$$

In the above development, the coefficients of j, $\frac{1}{1\cdot 2}j^2$, ..., $\frac{1}{n!}j^n$ are called the first, second, ..., n^{th} emanents of $\phi(x_1, x_2)$. The nth emanent may be symbolically represented by

$$\left(x_1'\frac{\delta}{\delta x_1} + x_2'\frac{\delta}{\delta x_2}\right)^n\phi.$$

It may readily be shown that these emanents are covariants of the function $\phi(x_1, x_2)$.

The reader will find no difficulty in extending the above to any number of variables.

136. Let us assume two sets each of n variables having the relation

$$u_1x_1 + u_2x_2 + \cdots + u_nx_n = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If this quantic be transformed by means of the linear substitutions

$$\left.\begin{aligned} x_1 &= a_{11}x'_1 + a_{12}x'_2 + \cdots + a_{1n}x'_n, \\ x_2 &= a_{21}x'_1 + a_{22}x'_2 + \cdots + a_{2n}x'_n, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n &= a_{n1}x'_1 + a_{n2}x'_2 + \cdots + a_{nn}x'_n, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

it will become

$$(a_{11}u_1+a_{21}u_2+\cdots+a_{n1}u_n)x'_1+(a_{12}u_1+a_{22}u_2+\cdots+a_{n2}u_n)x'_2$$
$$+\cdots+(a_{1n}u_1+a_{2n}u_2+\cdots+a_{nn}u_n)x'_n=0. \quad . \quad . \quad (3)$$

Writing

$$\left.\begin{array}{l} a_{11}u_1+a_{21}u_2+\cdots+a_{n1}u_n=u'_1, \\ a_{12}u_1+a_{22}u_2+\cdots+a_{n2}u_n=u'_2, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ a_{1n}u_1+a_{2n}u_2+\cdots+a_{nn}u_n=u'_n, \end{array}\right\} \quad . \quad . \quad . \quad (4)$$

the function (3) becomes

$$u'_1x'_1+u'_2x'_2+\cdots+u'_nx'_n=0, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is of the same form as the original quantic.

Solving Equations (4) for $u_1, u_2, \cdots, u_n$, representing the determinant of the system by δ and the co-factor of a_{ik} by A_{ik}, gives

$$\left.\begin{array}{l} \delta\cdot u_1=A_{11}u'_1+A_{12}u'_2+\cdots+A_{1n}u'_n, \\ \delta\cdot u_2=A_{21}u'_1+A_{22}u'_2+\cdots+A_{2n}u'_n, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \delta\cdot u_n=A_{n1}u'_1+A_{n2}u'_2+\cdots+A_{nn}u'_n. \end{array}\right\} \quad . \quad . \quad (6)$$

If, then, the quantic (1) be subjected to a linear substitution which leaves its form unaltered, the elements of the modulus of substitution for one set of variables, $u_1, u_2, \ldots, u_n$, are the co-factors of the corresponding elements of the modulus for the other set, $x_1, x_2, \ldots, x_n$.

Two sets of variables related in this manner are said to be **contragredient** or **reciprocal.**

NOTE. — The geometrical meaning of the above is that, in changing to a new system of reference, the triangular co-ordinates of a line are contragredient to the trilinear co-ordinates of a point.

In Article 129, the semi-differential coefficients q_1, q_2, ..., q_n are contragredient to x_1, x_2, ..., x_n, it being remembered that $a_{ik} \equiv a_{ki}$.

137. Resuming the equation

$$u_1x_1 + u_2x_2 + \cdots + u_nx_n = 0, \qquad (1)$$

let

$$\left.\begin{array}{l} u_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n, \end{array}\right\} \qquad (2)$$

where $a_{ik} = a_{ki}$.

Substituting these values of u_1, ..., u_n in (1), gives

$$a_{11}x_1^2 + a_{22}x_2^2 + \cdots + 2\,a_{12}x_1x_2 + 2\,a_{13}x_1x_3 + \cdots$$
$$+ 2\,a_{23}x_2x_3 + \cdots \equiv \Sigma\Sigma a_{ik}x_ix_k \equiv q.$$

Now, Equations (2) show that $x_1, \ldots, x$ are contragredient to u_1, ..., u_n. Hence, substituting in (1) the values of $x_1, \ldots, x_n$ expressed in terms of $u_1, \ldots, u_n$ gives

$$A_{11}u_1^2 + A_{22}u_2^2 + \cdots + 2\,A_{12}u_1u_2 + 2\,A_{13}u_1u_3 + \cdots$$
$$+ 2\,A_{23}u_2u_3 + \cdots \equiv \Sigma\Sigma A_{ik}u_iu_k \equiv Q.$$

The quadric Q is said to be reciprocal to q. It may be written in the form (Art. 38)

$$Q \equiv - \begin{vmatrix} a_{11} \cdots a_{1n} & u_1 \\ \cdot \quad \cdot \quad \cdot & \cdot \\ a_{n1} \cdots a_{nn} & u_n \\ u_1 \;\cdots\; u_n & 0 \end{vmatrix},$$

a form introduced by Hesse.

NOTE. — Geometrically, if the two quadrics are ternary, $q=0$ is the trilinear, and $Q=0$ the triangular equation of a conic. Equation (1) represents a tangent to $q=0$, or a point on $Q=0$.

If

$$\delta \equiv \begin{vmatrix} a_{11} \cdots a_{1n} \\ \cdot \quad \cdot \quad \cdot \\ a_{n1} \ldots a_{nn} \end{vmatrix} = 0,$$

we also have

$$\Delta \equiv \begin{vmatrix} A_{11} \cdots A_{1n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ A_{n1} \ldots A_{nn} \end{vmatrix} = 0$$

(Art. 60); that is, if q is composite, so also is Q.

But further, if $\delta = 0$, we have, by applying Article 62 to the symmetrical determinant δ,

$$A_{ik}^2 = A_{ii}A_{kk};$$

hence, using an obvious notation,

$$\begin{aligned} Q &= \Sigma\Sigma A_{ik}u_iu_k \\ &= \Sigma\Sigma(A_{ii}A_{kk})^{\frac{1}{2}}u_iu_k \\ &= (\Sigma A_{ii}^{\frac{1}{2}}u_i)^2; \end{aligned}$$

that is,

If the discriminant of a quadric vanishes, the reciprocal quadric is a perfect square.

(See Ex. 24, after Art. 50.)

EXAMPLES.

1. Write the discriminant and the Hessian of the ternary cubic.

2. Find the values of k for which the quadric

$$(4x^2 + 3z^2 - 5yz + 3xz + 2xy) \\ + k(x^2 + 3y^2 - yz + 5xz + 3xy) = 0$$

is resolvable into linear factors; that is, represents a pair of right lines.

Write, if possible, the square root of the reciprocal of each of the following quadrics:

3. $3x^2 - 6y^2 - 5z^2 + 13yz - 14xz + 7xy = 0.$

4. $x^2 + 3xy - xz = 0.$

5. $wx - wz - x^2 + xy + xz - yz = 0.$

6. $a^2x^2 - 2abxy + b^2y^2 = 0.$

7. $x^2 - y^2 + z^2 - 4yz + 6xz - 2xy = 0.$

8. $2xy - y^2 - 6xz + 3yz = 0.$

9. Show that the substitutions

$$x = x' \cos\alpha - y' \sin\alpha,\ y = x' \sin\alpha + y' \cos\alpha,$$

are orthogonal.

10. Show that the substitutions

$$x = x' \cos\alpha_1 + y' \cos\beta_1 + z' \cos\gamma_1,$$
$$y = x' \cos\alpha_2 + y' \cos\beta_2 + z' \cos\gamma_2,$$
$$z = x' \cos\alpha_3 + y' \cos\beta_3 + z' \cos\gamma_3,$$

are orthogonal; $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ being direction angles in Cartesian co-ordinates.

11. Write the coefficients for the quatenary orthogonal transformation.

12. Prove, by means of Equations (3), or (5), of Article 133, that any element of the modulus of an orthogonal substitution is equal to plus or minus its co-factor.

13. Prove that any minor of the modulus of an orthogonal substitution is equal to plus or minus its complementary co-factor.

14. Show that the emanents of a quantic are all of them, in general, of the same degree as the original quantic.

15. If the quadric

$$(a, b, c, f, g, h \between x, y, z)$$

be transformed by the linear substitutions

$$\begin{aligned} x &= -x' + y' + z', \\ y &= x' - 2y' + z', \\ z &= x' + y' - 3z', \end{aligned}$$

write the corresponding substitutions for the reciprocal quadric

$$(A, B, C, F, G, H \between u, v, w).$$

51930110R00141

Made in the USA
Lexington, KY
10 May 2016

From the PREFACE:

THE aim of the author of the present work has been to develop the Theory of Determinants in the simplest possible manner. Great care has been taken to introduce the subject in such a way that any reader having an acquaintance with the principles of elementary Algebra can intelligently follow this development from the beginning. The last two chapters must be omitted by the student who is not familiar with the Calculus, and the same is to be said in reference to some few of the preceding articles; but in no case will the continuity of the course be affected by such omissions.

No attempt has been made to apply the theory to Analytical Geometry, though a few of its more important applications to Algebra have been included. The reader familiar with geometrical analysis will be interested in giving to the greater number of these applications, as also to many of the examples, their geometrical interpretations.

The earlier the student, in his mathematical course, is made familiar with the notation and methods of Determinants, the earlier will he be prepared to appreciate the wonderful symmetry and generality so characteristic of the various modern developments in mathematics. In consideration of the limited time available for the study of such topics in the ordinary college course, the attempt has been made to render the book as readable as possible, rather than to prepare a drill book. However, it is hoped that the student who solves the two or three hundred examples proposed may thereby receive valuable mental discipline....

9781502300126
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